CO 330, LECTURE 25 SUMMARY

FALL 2017

Summary

After revisiting the labelled product from last time (with an example), we defined some more combinatorial operations.

Definition 1. *Let $\mathcal{C}$ be a labelled combinatorial class. Then define the combinatorial class of sequences of length k of elements of $\mathcal{C}$*

$$\textsc{Seq}_k(\mathcal{C}) = \underbrace{\mathcal{C} \star \cdots \star \mathcal{C}}_{k \text{ times}}$$

for $k \geq 1$ and $\textsc{Seq}_0(\mathcal{C}) = \mathcal{E}$. If $\mathcal{C}_0 = \emptyset$ then the combinatorial class of sequences of any length of elements of $\mathcal{C}$ is

$$\textsc{Seq}(\mathcal{C}) = \bigcup_{k \geq 0} \textsc{Seq}_k(\mathcal{C})$$

Then by the same argument as in the unlabelled case (but using the labelled product result from last time) we get

Proposition 2. *Let $\mathcal{C}$ be a labelled combinatorial class with $\textsc{Seq}_0 = \emptyset$ and let $\mathcal{A} = \textsc{Seq}(\mathcal{C}$. Then the exponential generating function of $\mathcal{A}$ is*

$$A(x) = \frac{1}{1 - C(x)}$$

This now gives us the nice operations we had in the unlabelled case, but in the labelled case we have more nice operations (these also have unlabelled analogues but they are more complicated; see assignment 7).

Definition 3. *Let $\mathcal{C}$ be a labelled combinatorial class. The labelled combinatorial class of k-element sets whose elements are from $\mathcal{C}$ is*

$$\textsc{Set}_k(\mathcal{C})$$

and the labelled combinatorial class of all finite sets of elements of $\mathcal{C}$ is

$$\textsc{Set}(\mathcal{C}) = \bigcup_{k \geq 0} \textsc{Set}_k(\mathcal{C})$$

Since the labelling forces everything to be distinct you can view k-element sets as k-element lists modulo permutation of the elements of the list, that is

$$\textsc{Set}_k(\mathcal{C}) = \textsc{Seq}_k(\mathcal{C})/P_k$$

where P_k is the permutation group on k elements. That's the formulation that gives us the following proposition.

Proposition 4. *Let $\mathcal{C}$ be a labelled combinatorial class, let $\mathcal{A} = \textsc{Set}_k(\mathcal{C})$ and let $\mathcal{B} = \textsc{Set}(\mathcal{C})$. Then the exponential generating functions are*

$$A(x) = \frac{C(x)^k}{k!} \qquad B(x) = \exp(C(x))$$

The proof of the first part is via the observation above which you could rephrase as a bijection

$$\textsc{Set}_k(\mathcal{C})P_k = \textsc{Seq}_k(\mathcal{C})$$

giving

$$A(x)k! = C(x)^k$$

Then summing

$$B(x) = \sum_{k\geq 0} C(x)^k/k! = \exp(C(x))$$

In a very similar way we can define cycles of combinatorial objects.

Definition 5. *Let $\mathcal{C}$ be a labelled combinatorial class. The labelled combinatorial class of k-element cycles of elements of $\mathcal{C}$ is*

$$\textsc{Cyc}_k(\mathcal{C})$$

and the labelled combinatorial class of all cycles of elements of $\mathcal{C}$ is

$$\textsc{Cyc}(\mathcal{C}) = \bigcup_{k\geq 1} \textsc{Cyc}_k(\mathcal{C})$$

Note that we don't allow cycles of length 0. Again, since the labelling forces everything to be distinct you can view k-element cycles as k-element lists modulo cyclic permutations, that is

$$\textsc{Cyc}_k(\mathcal{C}) = \textsc{Seq}_k(\mathcal{C})/C_k$$

where C_k is the cyclic group on k elements. Again that lets us calculate the exponential generating functions as follows.

Proposition 6. *Let $\mathcal{C}$ be a labelled combinatorial class, let $\mathcal{A} = \textsc{Cyc}_k(\mathcal{C})$ and let $\mathcal{B} = \textsc{Cyc}(\mathcal{C})$. Then the exponential generating functions are*

$$A(x) = \frac{C(x)^k}{k} \qquad B(x) = \log\left(\frac{1}{1 - C(x)}\right)$$

The proof is essentially the same as the previous proposition but with the cyclic group action in place of the permutation action.

References

We are continuing in the course notes chapter 11 or `http://people.math.sfu.ca/~kyeats/teaching/math343/labelled.pdf`. The course notes are more rigorous, the other notes less so. If you want more details along the lines of the other notes see Flajolet and Sedgewick's "Analytic Combinatorics".