CO 330, LECTURE 24 SUMMARY

FALL 2017

Summary

First, note that I am just going to use $C(x)$ for the exponential generating function of the labelled class $\mathcal{C}$ (and likewise for other letters), because that is consistent with both the class notes and the other notes (unlike what I did last time. You will have to use context to distinguish the ordinary and exponential generating functions based on whether we are working win an unlabelled or a labelled class.

Last time we noticed that if we think of permutations as labelled objects then we get the exponential generating function

$$P(x) = \sum_{\sigma \in \mathcal{P}} \frac{x^{|\sigma|}}{|\sigma|!} = \sum_{n \geq 0} \frac{n! x^n}{n!} = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

We can also do cycles $\mathcal{C}$ in this kind of direct counting way because the number of cycles on $\{1, 2, \ldots, n\}$ is $(n-1)!$ (by thinking of the n cyclic shifts of each permutation written as a word, or by thinking of fixing 1 as a start to the cycle and freely permuting the rest.) Therefore

$$C(x) = \sum_{c \in \mathcal{C}} \frac{x^{|c|}}{|c|!} = \sum_{n \geq 1} \frac{(n-1)! x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$$

Two things to take away from this. First this expansion of log will come back so it is worth remembering. Second $\exp(C(x)) = P(x)$ which is a special case of a much more general phenomenon which we'll return to next class.

Next we moved on to labelled specifications and labelled operations. We started with the stuff that works the same as in the unlabelled case:

Definition 1. $\mathcal{E} = \{\epsilon\}$ *which we can view as a labelled class by observing that* ϵ *has size* 0 *and so is uniquely labelled by* $\emptyset$ *by doing nothing.*

The exponential generating function of $\mathcal{E}$ is

$$E(x) = 1$$

Definition 2. $\mathcal{Z} = \{①\}$ *which is a labelled class with a unique element of size (and this element has a unique labelling by* 1*)*

The exponential generating function of $\mathcal{Z}$ is

$$Z(x) = x$$

Union of classes works just as in the unlabelled case: Let $\mathcal{A}$ and $\mathcal{B}$ be labelled classes with $\mathcal{A} \cap \mathcal{B} = \emptyset$, then every element of $\mathcal{A} \cup \mathcal{B}$ is either an element of $\mathcal{A}$ or $\mathcal{B}$ and so either way is correctly labelled.

The exponential generating functions add for the same reason as in the unlabelled case: Let $\mathcal{A}$ and $\mathcal{B}$ be labelled classes with $\mathcal{A} \cap \mathcal{B} = \emptyset$ and let $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. Then

$$C(x) = \sum_{c \in \mathcal{A} \cup \mathcal{B}} \frac{x^{|c|}}{|c|!} = \sum_{c \in \mathcal{A}} \frac{x^{|c|}}{|c|!} + \sum_{c \in \mathcal{B}} \frac{x^{|c|}}{|c|!} = A(x) + B(x)$$

However, product needs some more thought because if we just take an ordered pair of an element of $\mathcal{A}$ and an element of $\mathcal{B}$ then we'll get multiple copies of labels: the pair (a, b) will be labelled with $\{1, \ldots, |a|\}$ and $\{1, \ldots, |b|\}$ rather than with $\{1, \ldots, |a| + |b|\}$. In order to fix this we need to think a little more rigorously about labellings and also finally define labelled classes (at least as far as we will use them; the course notes make a more general definition)

Definition 3. *Let $\mathcal{A}$ be an unlabelled combinatorial class built by a specification out of $\mathcal{E}$, $\mathcal{Z}$, and admissible operations.*

- *For $a \in \mathcal{A}$, $|a|$ is the number of $\mathcal{Z}$ making up a; we call these $\mathcal{Z}$ the* ***atoms*** *of a.*
- *A* ***labelling*** *of $a \in \mathcal{A}$ is a bijection from the atoms of a to $\{1, \ldots, |a|\}$.*
- *A* ***weak labelling*** *of $a \in \mathcal{A}$ is a one-to-one map from a to $\{1, \ldots, m\}$.*

A ***labelled combinatorial class*** *$\mathcal{C}$ is a set of pairs (a, f) where $a \in \mathcal{A}$ for some $\mathcal{A}$ as above and f is a labelling of a. Additionally the size of (a, f) is the size of a in $\mathcal{A}$, and there are finitely many elements of $\mathcal{C}$ of any given size.*

To make the product work we need to be able to shift labellings.

Definition 4. *Let a be a labelled object.*

- *An* ***order preserving relabelling*** *of a is a weak labelling of a such that for any two atoms z_1 and z_2 of a $z_1 < z_2$ in the original ordering of a iff $z_1 < z_2$ in the new weak labelling.*
- *Given $S \subseteq \{1, \ldots, m\}$ with $|S| = |a|$ there is a unique order preserving relabelling of a with the elements of S. Write this as a_S.*

Now to build the product we just need to pick how to assign the elements of the bigger set of labels and then relabel. Specifically:

Definition 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be labelled combinatorial classes. Then*

$$\mathcal{A} \star \mathcal{B} = \bigcup_{n \geq 0} \left(\bigcup_{S \subseteq \{1, \ldots, n\}} \left\{ (a_S, b_{\{1, \ldots, n\} - S}) : a \in \mathcal{A}_{|S|}, b \in \mathcal{B}_{n - |S|} \right\} \right)$$

Note that we use $\star$ (or $*$) for this product rather than $\times$ so be careful with your notation. Note also that all the unions in the definition are disjoint. Also $\mathcal{A} \star \mathcal{B}$ really is a labelled combinatorial class because $(a_S, b_{\{1, \ldots, n\} - S})$ consists of a weak labelling of a and a weak labelled of b, but with disjoint labels and where the labels together give all of $\{1, \ldots, n\}$ so we can view it as a labelling of (a, b). Thus $\mathcal{A} \star \mathcal{B}$ is a set of labellings of elements of the unlabelled product.

The labelled product is the right definition because the exponential generating functions multiply. We did this at the very end of class but it was too rushed so we'll come back to it on Friday. Here's the computation: with notation as above

$$\begin{aligned}
(A\star B)(x) &= \sum_{n\geq 0}\sum_{S\subseteq\{1,\dots,n\}}\sum_{a_S,b_{\{1,\dots,n\}-S}} \frac{x^{|a|+|b|}}{(|a|+|b|)!} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\sum_{\substack{S\subseteq\{1,\dots,n\}\\ |S|=k}}\sum_{a_S,b_{\{1,\dots,n\}-S}} \frac{x^n}{n!} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k b_{n-k}x^n}{n!} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{a_k b_{n-k}x^n}{n!} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\frac{a_k}{k!}\frac{b_{n-k}}{(n-k)!}x^n \\
&= \left(\sum_{n\geq 0}\frac{a_n x^n}{n!}\right)\left(\sum_{n\geq 0}\frac{b_n x^n}{n!}\right) \\
&= A(x)B(x)
\end{aligned}$$

References

We are continuing in the course notes chapter 11 or `http://people.math.sfu.ca/~kyeats/teaching/math343/labelled.pdf`. I am not presenting things in exactly the same way as either source, but closer to the second one.