CO 330, LECTURE 23 SUMMARY

FALL 2017

Summary

We discussed the bijection from last time. There was more than one way to do it, but perhaps the easiest way is to take the tree of height at most two, remove the root, obtaining a collection of subtrees (each of height at most one), and then the partition is the partition of sizes of those subtrees.

Next we started labelled combinatorial objects. We looked at labelled rooted trees, labelled graphs, and permutations as labellings of a line. So far our only definition of a labelled combinatorial object is as a pair (c, f) where c is a combinatorial object in the unlabelled sense and a bijection between the "atoms" (think vertices) of the c and the set $\{1, 2, \ldots, |c|\}$. We will make this more precise when we have labelled specifications set up, but basically the atoms will be whatever was built of $\mathcal{Z}$ in the specification.

For labelled classes we want to use the exponential generating function. (When you need to distinguish the generating functions we had before are called *ordinary generating functions*). Specifically, if $\mathcal{C}$ is a labelled combinatorial class then the *exponential generating function* is

$$\widehat{C}(x) = \sum_{c \in \mathcal{C}} \frac{x^{|c|}}{|c|!}$$

Note that I would usually just notate this by $C(x)$, same as for ordinary generating functions, and let the context (labelled or unlabelled) distinguish between which type of generating function we are working with, but perhaps since we are just beginning with exponential generating functions, it is worth having a different notation.

As an example if we think of permutations as labelled objects, call the class $\mathcal{P}$, then

$$\widehat{P}(x) = \sum_{\sigma \in \mathcal{P}} \frac{x^{|\sigma|}}{|\sigma|!} = \sum_{n \geq 0} \frac{n! x^n}{n!} = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

which is a nice function.

Why exponential generating functions for labelled objects? First (and most importantly) they behave appropriately with the labelled constructions that we will develop. Second, dividing by $n!$ serves to scale by the maximum number of possible labellings. Third, they make it more likely that analytic techniques will be available (that is that the series will converge and not just be formal) in view of the larger number of labelled objects. This last one is related to the analytic notion of the Borel transform.

References

Along with the course notes chapter 11, take a look at `http://people.math.sfu.ca/~kyeats/teaching/math343/labelled.pdf`. The perspectives are somewhat different though the content is essentially the same. Those other notes are based on Flajolet and Sedgewick's book “Analytic Combinatorics” which is quite comprehensive but also rather practical in outlook. The course notes’ perspective is from the Joyal school of combinatorial species; this is a more rigorous but also heavier approach.