CO 330, LECTURE 19 SUMMARY

FALL 2017

Summary

The first thing we did today is go over the midterm. After that we went through the second decomposition which you'd thought about last time but we hadn't had time to go over. After that we began the proof of Euler's pentagonal number theorem:

$$\prod_{j=1}^{\infty}(1-x^j) = \sum_{h=-\infty}^{\infty} (-1)^h x^{h(3h-1)/2}$$

We did the mostly algebraic initial part of the proof and the combinatorial part was left until next time. Specifically, by series manipulation we showed that it is equivalent to show

$$\prod_{j=1}^{\infty}(1-x^j) = 1 + \sum_{h=1}^{\infty}(-1)^h \left(x^{h(3h-1)/2} + x^{h(3h+1)/2}\right)$$

Then we observed that if $\mathcal{D}$ is the combinatorial class of partitions with distinct parts then

$$\begin{aligned}
\prod_{j=1}^{\infty}(1-x^j) &= D(x,-1) \\
&= \sum_{\lambda\in\mathcal{D}}(-1)^{k(\lambda)}x^{n(\lambda)} \\
&= \sum_{\substack{\lambda\in\mathcal{D} \\ k(\lambda) \text{ is even}}} x^{n(\lambda)} - \sum_{\substack{\lambda\in\mathcal{D} \\ k(\lambda) \text{ is odd}}} x^{n(\lambda)} \\
&= A(x) - B(x) \\
&= \sum_{n=0}^{\infty}(a_n - b_n)x^n
\end{aligned}$$

where $\mathcal{A}$ is the combinatorial class of partitions with distinct parts and with an even number of parts and $\mathcal{B}$ is the combinatorial class of partitions with distinct parts and with an odd number of parts. Observe $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$ – we are breaking up by whether the number of parts is even or odd, not by whether the parts themselves are even or odd. Also $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$.

Now it remains to show that a_n and b_n almost cancel, only leaving something when $n = h(3h-1)/2$ or $n = h(3h+1)/2$. That is what we'll do next time.

References

The midterm solutions are on the website (in announcements). The decomposition we started with is Theorem 9.17 in the notes. Euler's pentagonal number theorem is the first topic of chapter 10 in the course notes.