CO 330, LECTURE 17 SUMMARY

FALL 2017

Summary

Today we returned to partitions.

We observed by generating function manipulations that the generating function for partitions with odd parts is the same as the generating function for partitions with distinct parts. Consequently the number of partitions of n with odd parts is equal to the number of partitions of n with distinct parts.

Then we found a bijection to prove this. Here is one way to describe it: Let $\mathcal{O}$ be the class of partitions with odd parts and let $\mathcal{D}$ be the class of partitions with distinct parts. Define $\alpha : \mathcal{D} \to \mathcal{O}$ as follows:

Given $\lambda \in \mathcal{D}$, take the largest even part and split it into two equal parts of half the size, continue until there are no even parts; the outcome of this procedure is $\alpha(\lambda)$. Observe a few things about this process

- This process terminates because either the size of the largest even part strictly decreases or the number of copies of the largest even part strictly decreases.
- This process preserves the size.
- We would get the same result if we made a different choice as to which even part to split at each step as long as we continue until we have split them all. The reason for this is because if an even part of λ has size $2^k m$ with m odd then it is split until it becomes 2^k parts of size m regardless of the order. (Sometimes this bijection is described directly this way rather than by an algorithm.)

Now define $\beta : \mathcal{O} \to \mathcal{D}$ as follows:

Given $\lambda \in \mathcal{O}$, take the smallest pair of equal parts and join them together to form a single part of double the size, continue until there are no duplicate parts; the outcome of this procedure is $\beta(\lambda)$. Observe a few things about this process

- This process terminates because either the size of the smallest pair of duplicate parts strictly increases or the number of copies of the smallest duplicate part strictly decreases.
- This process preserves the size.
- We would get the same result if we made a different choice as to which pair of duplicate parts to join at each step as long as we continue until we have no duplicate parts. The resason for this is if you let j be the number of copies of part a then write j's binary expansion, $j = b_0 \cdot 2^0 + b_1 \cdot 2^1 + b_1 \cdot 2^2 + \cdots$, then no matter what order we join parts in we will get exactly one part of size $2^k j$ iff $b_k = 1$. (Again sometimes this bijection is described directly this way rather than by an algorithm.)

Finally the two processes as described reverse each other step by step and so they are mutually inverse giving our bijection.

Today's class closed with a question for you to think about: why are self-conjugate partitions and partitions with odd and distinct parts in bijection? We will come back to that next class.

References

The generating function proof of the fact that partitions with odd parts and partitions with distinct parts are equinumerous is on p88 of the course notes (see the previous page for notation). The book leaves the bijection we gave to the problems (chapter 9 question 10). If you look it up online you can find many presentations usually (or maybe always) using the powers of 2/binary expansion approach. Once you see what's going on that's a great way to make sure you are saying things rigorously, but I like the iterative algorithmic description as well because it is more intuitive, and in the end they are the same.

The question about self-conjugate partitions and partitions with odd and distinct parts is example 9.15 in the course notes. You will get more out of it if you try yourself before you read the construction.