CO 330, LECTURE 16 SUMMARY

FALL 2017

Summary

Today we reviewed what we have done so far in the course. Then I did some problems on the board by request. I didn't have time to cover all the requests. The two I didn't cover were: a q-binomial identity by subsets so as to use the q-binomial theorem, and a LIFT example with $F(u) \neq u$.

For the q-binomial example lets start with a binomial identity that we proved using subsets. How about

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$$

which is proved by subsets in example 3.2 of the notes. I'll freely use notation from there without further comment (so if this makes no sense go look there!). Now keep track of the sum of the subsets to bring in the q parameter. We get

$$\sum_{(S,T)\in\mathcal{A}} q^{\sum_{s\in S} s} q^{\sum_{t\in T} t} = \sum_{(P,Q)\in\mathcal{B}} q^{\sum_{p\in P} p} q^{\sum_{a\in Q} a}$$

and each of these we can evaluate by the q-binomial theorem

$$\begin{aligned} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_q &= \sum_{(S,T)\in\mathcal{A}} q^{\sum_{s\in S} s} q^{\sum_{t\in T} t} \\ &= \sum_{(P,Q)\in\mathcal{B}} q^{\sum_{p\in P} p} q^{\sum_{a\in Q} a} \\ &= q^{j(j+1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{(k-j)(k-j+1)/2} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q \end{aligned}$$

collecting all the powers of q together we get

$$q^{j(2k-j+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q = \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q$$

For an example of LIFT without $F(u) = u$, let's just cook one up because I don't have a really natural one on hand. Suppose I have k-ary trees in the sense we had binary trees (with k slots for children any of which can be filled by empty trees), so let that class be $\mathcal{T}$ and the generating function $T(x)$ satisfies

$$T(x) = x(1 + T(x))^k$$

Now, suppose we don't actually care about these trees but the objects we actually want are lists of an even number of such trees. So we actually care about $\textsc{Seq}_{\text{even}}(\mathcal{T}) = \textsc{Seq}(\mathcal{T}^2)$, or at the level of generating functions we care about

$$F(T(x))$$

where $F(u) = 1/(1-u^2)$. Now we can calculate $[x^n]F(T(x))$ using LIFT where F is playing the role of F, $T(x)$ is playing the role of $R(x)$ and $(1+u)^k$ is playing the role of $G(u)$. We get

$$\begin{aligned}
[x^n]F(T(x)) &= \frac{1}{n}[u^{n-1}]F'(u)G(u)^n \\
&= \frac{1}{n}[u^{n-1}]\frac{-2u}{(1-u^2)^2}(1+u)^{nk} \\
&= \frac{-2}{n}[u^{n-2}](1-u^2)^{-2}(1+u)^{nk} \\
&= \frac{-2}{n}\sum_{k=0}^{n-2}[u^k](1-u^2)^{-2}[u^{n-2-k}](1+u)^{nk} \\
&= \frac{-2}{n}\sum_{0\le \ell \le (n-2)/2}\binom{-2}{\ell}\binom{2n\ell}{n-2-2\ell} \text{ where } k=2\ell \\
&= \frac{-2}{n}\sum_{0\le \ell \le (n-2)/2}(-1)^{\ell}\binom{\ell+1}{\ell}\binom{2n\ell}{n-2-2\ell} \\
&= \frac{2}{n}\sum_{0\le \ell \le (n-2)/2}(-1)^{\ell+1}(\ell+1)\binom{2n\ell}{n-2-2\ell}
\end{aligned}$$

Well that wasn't too ugly given that I just made it up off the top of my head rather than finding a particularly nice example. I don't promise there aren't little calculation errors in there so do check the details yourself.

References

Probably the best reference is to look back over the previous summaries as they provide a summary of the course.