CO 330, LECTURE 15 SUMMARY

FALL 2017

Summary

Don't forget the midterm is on Friday, during class time but in MC2034. Wednesday we will do some review, so bring any questions you have with you.

Today we proved the master theorem for integer partitions:

Theorem 1. *For each $j \in \mathbb{Z}_{\geq 1}$ let $M_j \subseteq \mathbb{Z}_{\geq 0}$. Let $\mathcal{A}$ be the class of partitions λ such that the number of parts equal to j is in M_j for all $j \geq 1$. Then*

$$\sum_{\lambda \in \mathcal{A}} x^{n(\lambda)} y^{k(\lambda)} = \prod_{j=1}^{\infty} \left(\sum_{m \in M_j} x^{jm} y^m \right)$$

For example, for partitions with distinct parts each $M_j = \{0, 1\}$, while for partitions with only odd parts we have $M_j = \begin{cases} \{0\} & j \text{ even} \\ \mathbb{Z}_{\geq 0} & j \text{ odd} \end{cases}$. There are other examples in the course notes and we'll return to this (including more examples) after the midterm.

You might wonder about the validity of the infinite product. The key is that to calculate $[x^n]A(x, y)$, the products for $j > n$ can only contribute their first term, namely 1, since all other terms are of the form $x^{jm}y^m$ for $j > n$ and $m \geq$. This means that to calculate $[x^n]A(x, y)$ we only need a finite product of formal power series, as the remaining factors contribute 1 and so are essentially not there. The only possible issue is if the remaining terms can't contribute 1 because $0 \notin M_j$. In such a situation $[x^n]A(x, y)$ is just 0. Provided $0 \in M_j$ for all but finitely many j then we don't have this issue for sufficiently large n and so the product is valid (and if not then we can define the product to be 0).

There are two ways to look at the proof of the master theorem. The proof in the course notes (see p86,87) goes through the bijection to multiplicity vectors and then the proof proceeds by manipulating formal power series. The other approach is to observe that we can think of a partition as a sequence of 1s followed by a sequence of 2s followed by a sequence of 3s etc. For unrestricted partitions this gives the decomposition

$$\mathcal{Y} = \textsc{Seq}(\{1\}) \times \textsc{Seq}(\{2\}) \times \textsc{Seq}(\{3\}) \times \cdots$$

For the class $\mathcal{A}$ from the statement of the master theorem this gives

$$\mathcal{A} = \textsc{Seq}_{M_1}(\{1\}) \times \textsc{Seq}_{M_2}(\{2\}) \times \textsc{Seq}_{M_3}(\{3\}) \times \cdots$$

where $\textsc{Seq}_M(\mathcal{B})$ means the combinatorial class of sequences of elements of $\mathcal{B}$ where the number of elements in the sequence must be in M. Furthermore $n(\lambda)$ sums the parts and so each 1 in the first sequence contributes 1, each 2 in the second sequence contributes 2 and so on, while $k(\lambda)$ counts the parts and so each sequence just contributes its length. Translating this specifications into generating functions gives

$$A(x,y) = \sum_{\lambda \in \mathcal{A}} x^{n(\lambda)} y^{k(\lambda)} = \left(\sum_{m \in M_1} x^m y^m \right) \left(\sum_{m \in M_2} x^{2m} y^m \right) \left(\sum_{m \in M_3} x^{3m} y^m \right) \cdots$$

which is what we wanted.

We spent the second half of the class discussing "How to help students understand lectures in advanced mathematics" by Weber, Fukawa-Connelly, Mejía-Ramos, and Lew, which can be found at `http://www.ams.org/publications/journals/notices/201610/rnoti-p1190.pdf`.

References

This material continues chapter 9 in the course notes. The education paper can be found at `http://www.ams.org/publications/journals/notices/201610/rnoti-p1190.pdf`. The American Mathematical Society (AMS) notices, have short education articles most issues under the heading Doceamus. Unfortunately there doesn't seem to be a central index to them, but if you search for "Doceamus" and AMS notices you can find them.