# CO 330, LECTURE 13 SUMMARY

FALL 2017

## Summary

Today we finally finished the proof of the $q$-binomial theorem. The remaining work was in proving the following proposition:

**Proposition 1.**

$$\sum_{S \in \mathcal{B}(n,k)} q^{\sum_{s \in S} s} = q^{\frac{k(k+1)}{2}} \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

The idea of the proof is to use the bijection $\mathcal{S}_n \simeq \mathcal{B}(n,k) \times \mathcal{S}_k \times \mathcal{S}_{n-k}$ and then consider how the inversions of a permutation of $\{1, 2, \ldots, n\}$ break up through this bijection: the ones involving two indices $\leq k$ become inversions in the permutation in $\mathcal{S}_k$, the ones involving two indices above $k$ become inversions in the permutation in $\mathcal{S}_{n-k}$ and the ones which cross $k$ are counted by $\sum_{s \in S} s - k(k+1)/2$ where $S$ is the set in $\mathcal{B}(n,k)$. See the course notes for details.

This finally gives everything we need for the $q$-binomial theorem itself

**Theorem 2** ($q$-binomial theorem)**.**

$$(1 + xq)(1 + xq^2) \cdots (1 + xq^n) = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \frac{[n]!_q}{[k]!_q [n-k]!_q} x^k$$

We define the $q$-binomial coefficient to be

$$\frac{[n]!_q}{[k]!_q [n-k]!_q}$$

and denote it

$$\begin{bmatrix} n \\ k \end{bmatrix}_q .$$

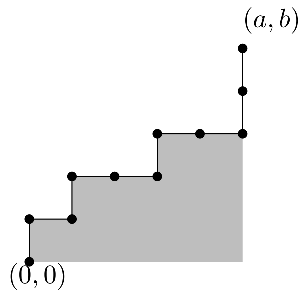
We next talked a little about the idea of $q$ analogues and finished up by proving

**Theorem 3.**

$$\sum_{P \in \mathcal{L}(a,b)} q^{\text{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

*where $\mathcal{L}(a,b)$ is the set of lattice paths beginning at $(0,0)$, ending at $(a,b)$ and using the steps $\uparrow$ and $\rightarrow$ and where the area of a lattice path is the area of the region bounded by it along with the segments $(0,0) - (a,0)$ and $(a,0) - (a,b)$.*

Here's an example




The area of this example is 11.

Now let's prove this theorem

*Proof.* Use the bijection $\mathcal{L}(a,b) \simeq \mathcal{B}(a+b,a)$ which takes a lattice path to the subset of $\{1,2,\ldots,a+b\}$ which index its $\to$ steps. In the case of the example above the subset would be $\{2,4,5,7,8\}$.

Given a lattice path $P$ let $S$ be the corresponding subset and write $S = \{s_1 < s_2 < \cdots < s_a\}$. In the case of the example above we'd have $s_1 = 2, s_2 = 4, s_3 = 5, s_4 = 7, s_5 = 8$. The area of the column topped by the step $\to$ corresponding to $s_i$ is the number of up steps before $s_i$ (since the $\to$ itself has length 1); this is the same as $s_i$ minus the number of right steps before $s_i$ including $s_i$ itself; this is $s_i - i$.

Therefore

$$\begin{aligned}\sum_{P\in\mathcal{L}(a,b)} q^{\mathrm{area}(P)} &= \sum_{S\in\mathcal{B}(a+b,a)} q^{\sum_{s\in S} s - \frac{a(a+1)}{2}} \\ &= \begin{bmatrix} a+b \\ a \end{bmatrix}_q q^{\frac{a(a+1)}{2}} q^{-\frac{a(a+1)}{2}} \\ &= \begin{bmatrix} a+b \\ a \end{bmatrix}_q\end{aligned}$$

□

If you didn't find the proof so clear in class look at the example (or make your own example).

## References

This material finishes chapter 5 of the course notes. The lattice path result is stated in the course notes as Theorem 5.8 but the proof is exercise 7 of chapter 5, which is why I gave more details on that proof in this summary.