CO 330, LECTURE 12 SUMMARY

FALL 2017

Summary

Today we looked at inversions in permutations. An inversion is a pair of indices where the corresponding entries occur in the wrong order. Formally,

Definition 1. *Let $a_1a_2\cdots a_n$ be a permutation of $\{1,2,\ldots,n\}$. An* inversion *is a pair (i,j) with $1 \le i < j \le n$ such that $a_i > a_j$.*

Then we proved

Proposition 2.
$$\sum_{\sigma\in\mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q[n-1]_q\cdots[2]_q[1]_q$$
where $\mathrm{inv}(\sigma)$ *is the number of inversions of σ and*
$$[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$$

The proof of this statement is by going through a the classical proof that there are $n!$ permutations very carefully while keeping track of the inversions.

$[k]_q$ is the q-analogue of the natural number k. Not surprisingly we also define
$$[n]!_q = [n]_q[n-1]_q\cdots[2]_q[1]_q$$
which is the q-factorial.

We ended by stating the next proposition

Proposition 3.
$$\sum_{S\in\mathcal{B}(n,k)} q^{\sum_{s\in S} s} = q^{\frac{k(k+1)}{2}}\frac{[n]!_q}{[k]!_q[n-k]!_q}$$

which we will also prove by revisiting the classical proof, but that's for next time.

References

This material continues chapter 5 of the course notes and again the classical proof can be found in chapter 2 of the course notes. If you're not sure about how to rigorously write down any of the details we left out the course notes are a very good place to look.