CO 330, LECTURE 11 SUMMARY

FALL 2017

Summary

Today we started to look at the q-binomial theorem. Everything we did was in parallel with things we could do for the classical binomial theorem.

First on the classical side we reminded ourselves that the usual binomial theorem is a result about the generating function of subsets of $\{1, 2, \ldots, n\}$ where the weight is the size of the subset. That is

$$\sum_{S \subseteq N_n} x^{|S|} = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where $N_n = \{1, 2, \ldots, n\}$. Thinking about it this way we can get a closed form using a bijection from S to its characteristic function (represented as a binary string): $S \mapsto b_1 b_2 \cdots b_n$ where $b_i = 0$ if $i \notin S$ and $b_i = 1$ if $i \in S$. Note that under this map $|S|$ becomes $b_1 + b_2 + \cdots + b_n$. Thus for the generating function we get

$$\begin{aligned}
\sum_{S \subseteq N_n} x^{|S|} &= \sum_{b_1 b_2 \cdots b_n \in \{0,1\}^n} x^{b_1 + b_2 + \cdots + b_n} \\
&= \left(\sum_{b_1 \in \{0,1\}} x^{b_1} \right) \left(\sum_{b_2 \in \{0,1\}} x^{b_2} \right) \cdots \left(\sum_{b_n \in \{0,1\}} x^{b_n} \right) \\
&= \underbrace{(1+x)(1+x)\cdots(1+x)}_{n \text{ times}} = (1+x)^n
\end{aligned}$$

Now for the q-analogue, lets do this all again with the weight of S being $(|S|, \sum_{s \in S} s)$. We again have the generating function

$$\sum_{S \subseteq N_n} x^{|S|} q^{\sum_{s \in S} s}$$

It is traditional to call the second parameter q. There are two questions now. How to find the coefficient of x^k, so what is the q-analogue of $\binom{n}{k}$ and what closed form we get, so what is the q-analogue of $(1+x)^n$.

For the closed form we can use the same bijection to the characteristic function. With a little thought we saw that $\sum_{s \in S} s$ becomes $b_1 + 2b_2 + 3b_3 + \cdots + nb_n$. Try some examples if you don't see that. Now for the generating function we get

$$\begin{aligned}\sum_{S\subseteq N_n} x^{|S|}q^{\sum_{s\in S} s} &= \sum_{b_1b_2\cdots b_n\in\{0,1\}^n} x^{b_1+b_2+\cdots+b_n}q^{b_1+2b_2+\cdots+nb_n}\\ &= \left(\sum_{b_1\in\{0,1\}} x^{b_1}q^{b_1}\right)\left(\sum_{b_2\in\{0,1\}} x^{b_2}q^{2b_2}\right)\cdots\left(\sum_{b_n\in\{0,1\}} x^{b_n}q^{nb_n}\right)\\ &= (1+x)(1+xq^2)(1+xq^3)\cdots(1+xq^n)\end{aligned}$$

To get the q-binomial theorem we still need to have an expression for the coefficient of x^k in $\sum_{S\subseteq N_n} x^{|S|}q^{\sum_{s\in S} s}$. It will be some kind of polynomial in q. To find it we again revisited the proof that subsets are counted by $\binom{n}{k}$ and we looked in detail at the pseudocode from the course notes. To get a q-analogue we need to know what q should count on the permutations. We observed that if S has big elements in it then that will cause big elements to appear early in the permutation. This is suggestive that it is important when big elements appear before littler elements in permutations. This gives the notion of *inversion* in a permutation which will do the job and is what we'll talk about next class.

References

This material is the beginning of chapter 5 of the course notes, combined with the revisited old material which can be found in chapter 2 of the course notes.