

Cointeractions of Some Word Bialgebras

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1 Introduction

Cointeractions are a new topic in abstract algebra that attracts wide interest from mathematicians around the world. Its inherent relation to combinatorial structures promises applications in many fields such as graph theory and quantum field theory in theoretical physics. For example, in 2008, Damien Calaque, Kurusch Ebrahimi-Fard, and Dominique Manchon demonstrated the cointeraction between two Hopf algebras on rooted trees[6]. In addition, Mohamed Ayadi and Dominique Manchon explore the cointeraction of two bialgebras from finite topologies in their 2020 paper[7]. In 2021, Dirk Kreimer and Karen Yeats constructed two Hopf algebras from Feynmann graphs in the quantum physics and proved their cointeraction[4]. This paper attempts to build up the work by Kreimer and Yeats and add the order structure on the edges of the graphs they were studying.

In this paper, we want to develop cointeractions on word bialgebra, specifically, the word bialgebras that show up in studying the fundamental cycles and order structure on the edges of finite graphs. In Section 2, we want to set up the algebraic foundation for bialgebras and discuss in detail the two classic word bialgebras, namely, the concatenation-deshuffle and shuffle-deconcatenation bialgebras. We also introduce the concept of grading and Hopf algebra, since many bialgebras that appear naturally in combinatorial structures are Hopf algebras. In Section 3, we construct the concatenation-deshuffle bialgebra on two alphabets which has a graph theoretical motivation. However, when trying to prove the cointeraction on the incidence structure of the concatenation-deshuffle bialgebra on two alphabets, we run into issues caused by words' rigid dependence on orders. We offer two approaches to resolve this issue and build up the correct cointeraction relation.

2 Algebraic Background

In this section, we will systematically build up the algebraic background for this paper. Most of the results can be found in [1][2][3]. However, we will supply most of the proofs and explain how they are related to the results of this paper.

We will first introduce the concepts of algebra and coalgebra that leads to the concept of bialgebra. Then we will discuss the concatenation-deshuffle and shuffle-deconcatenation bialgebras as two examples to illustrate how to construct and verify the properties of bialgebras. These two bialgebras will be related to the algebra structure that we are trying to construct in the next section. Next we will define commutativity and grading in bialgebras, since they will be important in finding the antipode that we need to construct a Hopf algebra structure. With all the tools, we will define Hopf algebras. In the final subsection, we introduce the incidence structure that we will utilize in the next section.

2.1 Algebra and Coalgebra

First we need to introduce the concept of algebra.

Definition 1. *An algebra A over a field K is a vector space over K with two linear maps, the multiplication map $m : A \otimes A \rightarrow A$ and the unit map $u : K \rightarrow A$ such that the following two diagrams commute.*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\text{Id} \otimes m} & A \otimes A \\ \downarrow m \otimes \text{Id} & & \downarrow m \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

$$\begin{array}{ccccc} K \otimes A & \xleftarrow{a \mapsto 1 \otimes a} & A & \xrightarrow{a \mapsto a \otimes 1} & A \otimes K \\ \downarrow u \otimes \text{Id} & & \downarrow \text{Id} & & \downarrow \text{Id} \otimes u \\ A \otimes A & \xrightarrow{m} & A & \xleftarrow{m} & A \otimes A \end{array}$$

The first diagram shows that the multiplication map m is associative. For any $a_1, a_2, a_3 \in A$, we have $m(m(a_1 \otimes a_2) \otimes a_3) = m(a_1 \otimes m(a_2 \otimes a_3))$. The second diagram shows that the algebra behaves like a ring, where $m(u(1) \otimes a) = m(a \otimes u(1)) = a$ for any $a \in A$, so $u(1)$ is the multiplicative identity in A .

Note that the multiplication map here is defined on $A \otimes A$ instead of the usual $A \times A$. This definition is valid because of the universal property of tensor products.

Theorem 1 (Universal Property of Tensor Product). *For any algebras A, B and a bilinear map $f : A \times A \rightarrow B$, there exists a unique linear map $\hat{f} : A \otimes A \rightarrow B$ such that $\hat{f} \circ r = f$. Here r is the bilinear map $r : A \otimes A \rightarrow A \otimes A$ such that $r(a_1, a_2) = a_1 \otimes a_2$ for any $a_1, a_2 \in A$.*

$$\begin{array}{ccc} A \times A & \xrightarrow{r} & A \otimes A \\ & \searrow f & \downarrow \hat{f} \\ & & B \end{array}$$

The proof of this universal property depends on the construction of the vector space $A \otimes A$ that we will not explore in this paper. From this point onward, we will use both definitions of multiplications interchangeably.

Tensor products tend to preserve some algebraic structures of factor. For example, it is easy to verify that if A is an algebra, then $A \otimes A$ is also an algebra. However, we will soon find out that we cannot just naively tensor the operations together to get the correct operation. We need to be careful when deriving properties of the tensor product from its factors.

Definition 2. A coalgebra C over a field K is a vector space over K with two linear maps, the coproduct $\Delta : C \rightarrow C \otimes C$, and the counit map $\epsilon : C \rightarrow K$ such that the following two diagrams commute.

$$\begin{array}{ccc}
 C \otimes C \otimes C & \xleftarrow{\text{Id} \otimes \Delta} & C \otimes C \\
 \Delta \otimes \text{Id} \uparrow & & \uparrow \Delta \\
 C \otimes C & \xleftarrow{\Delta} & C
 \end{array}$$

$$\begin{array}{ccccc}
 K \otimes C & \xrightarrow{k \otimes c \rightarrow kc} & C & \xleftarrow{c \otimes k \rightarrow kc} & C \otimes K \\
 \epsilon \otimes \text{Id} \uparrow & & \text{Id} \uparrow & & \text{Id} \otimes \epsilon \uparrow \\
 C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C
 \end{array}$$

The first diagram shows that the coproduct map Δ is co-associative, and the second diagram shows that the coalgebra behaves like a co-ring.

Next we introduce the notion of algebra homomorphism and coalgebra homomorphism.

Definition 3. Let A and B be algebras over K . Then the map $f : A \rightarrow B$ is an algebra homomorphism if the following two diagrams commute.

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 m_A \uparrow & & \uparrow m_B \\
 A \otimes A & \xrightarrow{f \otimes f} & B \otimes B
 \end{array}$$

$$\begin{array}{ccc}
 K & \xrightarrow{u_B} & B \\
 u_A \downarrow & \nearrow f & \\
 A & &
 \end{array}$$

This definition matches our understanding of ring homomorphism. The first diagram shows that the product of the map equals the map of the product, and the second diagram shows that f maps a unit in A to a unit in B .

Definition 4. Let C and D be coalgebras over K . Then the map $g : C \rightarrow D$ is a coalgebra homomorphism if the following two diagrams commute.

$$\begin{array}{ccc}
C & \xrightarrow{g} & D \\
\downarrow \Delta_C & & \downarrow \Delta_D \\
C \otimes C & \xrightarrow{g \otimes g} & D \otimes D
\end{array}$$

$$\begin{array}{ccc}
C & \xrightarrow{g} & D \\
\searrow \epsilon_C & & \downarrow \epsilon_D \\
& & K
\end{array}$$

Similar to our discussion above, the coalgebra homomorphism g matches our understanding of co-ring homomorphism.

Theorem 2. *If a vector space B is simultaneously an algebra and a coalgebra over K , then the product and unit are coalgebra homomorphisms if and only if the coproduct and counit are algebra homomorphisms.*

Proof. First we want to show the sufficiency. If the coproduct is an algebra homomorphism, then the following four diagrams commute. Note that the unit map from K to K is the identity map.

$$\begin{array}{ccc}
B & \xrightarrow{\Delta} & B \otimes B \\
m \uparrow & & m_{13,24} \uparrow \\
B \otimes B & \xrightarrow{\Delta \otimes \Delta} & B \otimes B \otimes B \otimes B
\end{array}$$

$$\begin{array}{ccc}
K & \xrightarrow{u_{B \otimes B}} & B \otimes B \\
u \downarrow & \nearrow \Delta & \\
B & &
\end{array}$$

$$\begin{array}{ccc}
B & \xrightarrow{\epsilon} & K \\
m \uparrow & & m_K \uparrow \\
B \otimes B & \xrightarrow{\epsilon \otimes \epsilon} & K \times K
\end{array}$$

$$\begin{array}{ccc}
K & \xrightarrow{\text{Id}} & K \\
u \downarrow & \nearrow \epsilon & \\
B & &
\end{array}$$

We introduce the map $m_{1,3,2,4} : A \otimes A \otimes A \otimes A \rightarrow A \otimes A \otimes A \otimes A$ such that

$$m_{1,3,2,4}(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_1 \otimes a_3 \otimes a_2 \otimes a_4,$$

and let $m_{13,24} = (m \otimes m) \circ m_{1,3,2,4}$. We introduce this map instead of using $m \otimes m$ directly because $m_{13,24}$ is the multiplication deduced with the universal property

of tensor product and the original multiplication map $\hat{m} : B \otimes B \times B \otimes B \rightarrow B \otimes B$ multiplies the first and third terms, and second and fourth terms together respectively. If we use $m \otimes m$ directly, then the first and second terms, and third and fourth terms are multiplied together respectively, which is not the multiplication map we would get from the universal property of tensor product.

Hence, we have

$$\begin{aligned}\Delta \circ m &= (m_{13,24}) \circ (\Delta \otimes \Delta) = (m \otimes m) \circ m_{1,3,2,4} \circ (\Delta \otimes \Delta), \\ \Delta \circ u &= u_{B \otimes B}, \\ \epsilon \circ m &= m_K \circ (\epsilon \otimes \epsilon), \\ \epsilon \circ u &= \text{Id}.\end{aligned}$$

We know that the counit map from K to K is the identity map. Let $\Delta_{13,24} = m_{1,3,2,4} \circ (\Delta \otimes \Delta)$. The following two diagrams commute.

$$\begin{array}{ccc} B \otimes B & \xrightarrow{m} & B \\ \downarrow \Delta_{13,24} & & \downarrow \Delta \\ B \otimes B \otimes B \otimes B & \xrightarrow{m \otimes m} & B \otimes B \end{array}$$

$$\begin{array}{ccc} K & \xrightarrow{u} & B \\ & \searrow \text{Id} & \downarrow \epsilon \\ & & K \end{array}$$

The map $m_{1,3,2,4}$ is also required in writing out the coproduct of $B \otimes B$. By the universal property of tensor product, we want the terms from first B to occupy the first and third positions and the terms from second B to occupy the second and fourth positions.

Now we look at the other two diagrams.

$$\begin{array}{ccc} B \otimes B & \xrightarrow{m} & B \\ & \searrow \epsilon_{B \otimes B} & \downarrow \epsilon \\ & & K \end{array}$$

$$\begin{array}{ccc} K & \xrightarrow{u} & B \\ \downarrow \Delta_K & & \downarrow \Delta \\ K \otimes K & \xrightarrow{u \otimes u} & B \otimes B \end{array}$$

Since the coalgebra $B \otimes B$ is the tensor product of the coalgebra B , we have $\epsilon_{B \otimes B} = m_K \circ (\epsilon \otimes \epsilon)$. Hence, we have $\epsilon \circ m = m_K \circ (\epsilon \otimes \epsilon) = \epsilon_{B \otimes B}$ and the diagram commutes.

Since K is a one-dimensional vector space over K with the basis $\{1\}$, we can see that $K \otimes K$ is also a one-dimensional vector space over K with the basis $\{1 \otimes 1\}$. By the definition of coalgebra, the coproduct Δ_K can only map 1 to $1 \otimes 1$, and by the universal property of vector spaces, Δ_K is defined over K . Since the algebra $B \otimes B$ is the tensor product of the algebra B , we have $u_{B \otimes B} = (u \otimes u) \circ \Delta_K$. Hence, we have $\Delta \circ u = u_{B \otimes B} = (u \otimes u) \circ \Delta_K$ and the diagram commutes.

Therefore, if the coproduct and counit are algebra homomorphisms, then the product and unit are coalgebra homomorphisms.

We note that sixth diagram is the same as the fourth diagram up to rotation. In addition, $\Delta \circ m = (m \otimes m) \circ \Delta_{13,24} = (m \otimes m) \circ m_{1,3,2,4} \circ (\Delta \otimes \Delta)$ is the same as $\Delta \circ m = (m_{13,24}) \circ (\Delta \otimes \Delta) = (m \otimes m) \circ m_{1,3,2,4} \circ (\Delta \otimes \Delta)$. By our previous discussion, we can see that the second diagram is the same as the eighth diagram and the third diagram is the same as the seventh diagram, so the necessity follows. \square

With the definitions of algebra, coalgebra and algebra homomorphism, we are able to define bialgebra.

Definition 5. *If the vector space B is both an algebra and a coalgebra over K and the coproduct and counit are algebra homomorphisms, then B is a bialgebra over K .*

2.2 Concatenation-Deshuffle and Shuffle-Deconcatenation Bialgebras

We want to introduce the two bialgebras that we are discussing in this paper. Let \mathcal{W} be the set of all words over an alphabet Ω , and let $W = \text{span}_K \mathcal{W}$ be the vector space of formal linear combinations of words. By the definition of W , we can see that \mathcal{W} is a basis of W .

First we can define the concatenation-deshuffle bialgebra. It suffices to define the multiplication map \hat{m} over the basis $\mathcal{W} \otimes \mathcal{W}$. For any $w_1 \otimes w_2 \in \mathcal{W} \otimes \mathcal{W}$, let $\hat{m}(w_1 \otimes w_2)$ be the concatenation of w_1 and w_2 and i be the inclusion map. By the universal property of vector spaces, there exists a unique linear map m which we define as the multiplication for the bialgebra. We call this multiplication concatenation.

$$\begin{array}{ccc} \mathcal{W} \otimes \mathcal{W} & \xrightarrow{i} & W \otimes W \\ & \searrow \hat{m} & \downarrow m \\ & & W \end{array}$$

Similarly, we define coproduct $\hat{\Delta}$ over the the basis \mathcal{W} . For any non-empty word $a_1 \cdots a_n \in \mathcal{W}$, where $a_1, \dots, a_n \in \Omega, n \in \mathbb{Z}_+$, we let the $\hat{\Delta}$ be the deshuffle map such that

$$\hat{\Delta}(a_1 \cdots a_n) = \sum_{I \subset \{1, \dots, n\}} a_I \otimes a_{\{1, \dots, n\} \setminus I},$$

where a_I is a subword of $a_1 \cdots a_n$ consisting of letters indexed by I . We define $\hat{\Delta}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}$, where \mathcal{E} is the empty word. For example, we have

$$\hat{\Delta}(aba) = \mathcal{E} \otimes aba + a \otimes ba + b \otimes aa + a \otimes ab + ab \otimes a + aa \otimes b + ba \otimes a + aba \otimes \mathcal{E},$$

where $a, b \in \Omega$. By the universal property of vector spaces, there exists a unique linear map Δ which we define as the coproduct for the bialgebra. We call this coproduct deshuffle.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{i} & W \\ & \searrow \hat{\Delta} & \downarrow \Delta \\ & & W \otimes W \end{array}$$

For any $k \in K$, we can define the unit map $u(k) = k\mathcal{E}$ where \mathcal{E} is the empty word. We define the counit $\hat{\epsilon}$ over the basis \mathcal{W} . Let $\hat{\epsilon}(\mathcal{E}) = 1$ and $\hat{\epsilon}(w) = 0$ for any non-empty word $w \in \mathcal{W}$. By the universal property of vector spaces, there exists a unique linear map ϵ which we define as the counit for the bialgebra.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{i} & W \\ & \searrow \hat{\epsilon} & \downarrow \epsilon \\ & & K \end{array}$$

We need to verify that our W constructed in the way described above is indeed a bialgebra. By the universal property of vector spaces, we only need to verify that the diagrams in the definition of bialgebras commute over the basis of W .

For any $w_1, w_2, w_3 \in \mathcal{W}$, we can see that m is associative, since

$$m(m(w_1 \otimes w_2) \otimes w_3) = \overline{w_1 w_2 w_3} = m(w_1 \otimes m(w_2 \otimes w_3)),$$

where $\overline{w_1 w_2 w_3}$ is the concatenation of w_1, w_2, w_3 in that order.

For any $w \in \mathcal{W}$, we have $\text{Id} \otimes u(w \otimes 1) = w \otimes \mathcal{E}$, so $m(w \otimes \mathcal{E}) = w$. Similarly, we have $u \otimes \text{Id}(1 \otimes w) = \mathcal{E} \otimes w$, so $m(\mathcal{E} \otimes w) = w$. Both diagrams in the definition commute, so W is an algebra.

If $w = \mathcal{E}$, we have $(\Delta \circ \text{Id}) \circ \Delta(w) = \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E} = (\text{Id} \circ \Delta) \circ \Delta(w)$. For any $w = a_1 \cdots a_n \in \mathcal{W}$ where $n \in \mathbb{Z}_+$, we have

$$(\Delta \circ \text{Id}) \circ \Delta(w) = \sum_{I, J, K \text{ disjoint}, I \cup J \cup K = \{1, \dots, n\}} a_I \otimes a_J \otimes a_K = (\text{Id} \circ \Delta) \circ \Delta(w),$$

so Δ is co-associative.

For any $w \in \mathcal{W}$, we have $(\text{Id} \otimes \epsilon) \circ \Delta(w) = w \otimes 1$, since all summands other than $w \otimes \mathcal{E}$ are mapped to 0 because ϵ maps all non-empty words to 0. We have $1 \cdot w = w = \text{Id}(w)$. Following the same argument, we have $(\epsilon \otimes \text{Id}) \circ \Delta(w) = 1 \otimes w$, and $1 \cdot w = w = \text{Id}(w)$. Both diagrams in the definition commute, so W is a coalgebra.

For any $k \in K$, we have $u(k) = k\mathcal{E}$, so $\epsilon \circ u(k) = k\epsilon(\mathcal{E}) = k = \text{Id}(k)$.

For any $w_1, w_2 \in \mathcal{W}$, if w_1 or w_2 is empty (without loss of generality, we can assume that $w_1 = \mathcal{E}$), then

$$\epsilon \circ m(w_1 \otimes w_2) = \epsilon(w_2) = 1 \cdot \epsilon(w_2) = \epsilon(w_1)\epsilon(w_2) = m_K \circ (\epsilon \otimes \epsilon)(w_1 \otimes w_2).$$

If w_1 and w_2 are non-empty, then $m(w_1 \otimes w_2)$ is also non-empty, so

$$\epsilon \circ m(w_1 \otimes w_2) = 0 = 0 \cdot 0 = \epsilon(w_1)\epsilon(w_2) = m_K \circ (\epsilon \otimes \epsilon)(w_1 \otimes w_2).$$

We conclude that the counit ϵ is an algebra homomorphism.

For any $k \in K$, we have $\Delta \circ u(k) = k\Delta(\mathcal{E}) = k\mathcal{E} \otimes \mathcal{E} = u_{W \otimes W}(k)$.

For any $w_1, w_2 \in \mathcal{W}$, if w_1 or w_2 is empty (without loss of generality, we can assume that $w_1 = \mathcal{E}$), then $\Delta \circ m(w_1 \otimes w_2) = \Delta(w_2)$. We can see that $\Delta \otimes \Delta(w_1 \otimes w_2) = \mathcal{E} \otimes \mathcal{E} \otimes \Delta(w_2)$ and $m_{13,24}(\mathcal{E} \otimes \mathcal{E} \otimes \Delta(w_2)) = \Delta(w_2) = \Delta \circ m(w_1 \otimes w_2)$. For any $w_1 = a_1 \cdots a_k, w_2 = a_{k+1} \cdots a_{k+l} \in \mathcal{W}$ where $k, l \in \mathbb{Z}_+$, we have

$$\Delta \circ m(w_1 \otimes w_2) = \sum_{I \in \{1, \dots, k+l\}} a_I \otimes a_{\{1, \dots, k+l\} \setminus I},$$

and

$$\begin{aligned} & m_{13,24} \circ (\Delta \otimes \Delta)(w_1 \otimes w_2) \\ &= m_{13,24} \left(\sum_{I \in \{1, \dots, k\}} \sum_{J \in \{k+1, \dots, k+l\}} a_I \otimes a_{\{1, \dots, k\} \setminus I} \otimes a_J \otimes a_{\{k+1, \dots, k+l\} \setminus J} \right) \\ &= \sum_{I \in \{1, \dots, k\}, J \in \{k+1, \dots, k+l\}} a_{I \cup J} \otimes a_{\{1, \dots, k+l\} \setminus (I \cup J)} \\ &= \Delta \circ m(w_1 \otimes w_2) \end{aligned}$$

We conclude that the coproduct Δ is an algebra homomorphism. Therefore, W constructed on concatenation and deshuffle is indeed a bialgebra.

We can also define the shuffle-deconcatenation bialgebra structure on W . Similar to our discussion above, by the universal property of vector spaces, it suffices to define the multiplication map \hat{m} over the basis $\mathcal{W} \otimes \mathcal{W}$. For any $w_1, w_2 \in \mathcal{W}$, let $\hat{m}(w_1, w_2) = \text{sum of all words formed by shuffling } w_1 \text{ and } w_2$. If $w_1 = \mathcal{E}$, then $\hat{m}(w_1, w_2) = w_2$ and if $w_2 = \mathcal{E}$, then $\hat{m}(w_1, w_2) = w_1$. If $w_1 = u_1 \cdots u_k, w_2 = u_{k+1} \cdots u_{k+l} \in \mathcal{W}$ where $k, l \in \mathbb{Z}_+$, then

$$\hat{m}(w_1, w_2) = \sum_{\substack{\sigma \text{ permutes } \{1, \dots, k+l\} \\ \sigma^{-1}(1) < \dots < \sigma^{-1}(k) \\ \sigma^{-1}(k+1) < \dots < \sigma^{-1}(k+l)}} u_{\sigma(1)} \cdots u_{\sigma(k+l)},$$

. We denote the multiplication from the universal property of vector spaces as \sqcup and call it shuffle. Then we have the following examples,

$$uv \sqcup x = uvx + uxv + xuv,$$

$$\begin{aligned} uv \sqcup ux &= uvux + uvvx + uvvx + uuxv + uuxv + uxuv \\ &= uvux + 2uvvx + 2uuxv + uxuv. \end{aligned}$$

Next we define the coproduct over the basis \mathcal{W} . Let $\hat{\Delta}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}$. For any $w = u_1 \cdots u_k$ where $k \in \mathbb{Z}_+$, we have

$$\hat{\Delta}(w) = \mathcal{E} \otimes w + w \otimes \mathcal{E} + \sum_{i=1}^{k-1} u_1 \cdots u_i \otimes u_{i+1} \cdots u_k.$$

We call the coproduct from the universal property of vector spaces deconcatenation, and we have the following examples,

$$\Delta(aba) = \mathcal{E} \otimes aba + aba \otimes \mathcal{E} + a \otimes ba + ab \otimes a,$$

where $a, b \in \Omega$.

We use the same definitions of unit and counit in the concatenation-deshuffle bialgebra.

We also want to verify that shuffle and deconcatenation do introduce a bialgebra structure. For any $w_1, w_2, w_3 \in \mathcal{W}$, if any of the three word is empty, then we have $(w_1 \sqcup w_2) \sqcup w_3 = w_1 \sqcup (w_2 \sqcup w_3)$ since both of them equal to the shuffle product of the other two words. If $w_1 = u_1 \cdots u_j, w_2 = u_{j+1} \cdots u_{j+k}, w_3 = u_{j+k+1} \cdots u_{j+k+l} \in \mathcal{W}$ where $j, k, l \in \mathbb{Z}_+$, since the composition of two permutation is still a permutation, we have

$$(w_1 \sqcup w_2) \sqcup w_3 = \sum_{\substack{\sigma \text{ permutes } \{1, \dots, j+k+l\} \\ \sigma^{-1}(1) < \dots < \sigma^{-1}(j) \\ \sigma^{-1}(j+1) < \dots < \sigma^{-1}(j+k) \\ \sigma^{-1}(j+k+1) < \dots < \sigma^{-1}(j+k+l)}} u_{\sigma(1)} \cdots u_{\sigma(j+k+l)} = w_1 \sqcup (w_2 \sqcup w_3).$$

Hence, the shuffle product is associative.

For any $w \in \mathcal{W}$, we have $\text{Id} \otimes u(w \otimes 1) = w \otimes \mathcal{E}$, so $m(w \otimes \mathcal{E}) = w$. Similarly, we have $u \otimes \text{Id}(1 \otimes w) = \mathcal{E} \otimes w$, so $m(\mathcal{E} \otimes w) = w$. Both diagrams in the definition commute, so W is an algebra.

For any $w \in \mathcal{W}$, we have

$$(\Delta \otimes \text{Id}) \circ \Delta(w) = \sum_{w_1, w_2, w_3 \in \mathcal{W}, \overline{w_1 w_2 w_3} = w} w_1 \otimes w_2 \otimes w_3 = (\text{Id} \otimes \Delta) \circ \Delta(w),$$

so Δ is co-associative.

For any $w \in \mathcal{W}$, we have $(\text{Id} \otimes \epsilon) \circ \Delta(w) = w \otimes 1$, since all summands other than $w \otimes \mathcal{E}$ are mapped to 0 because ϵ maps all non-empty words to 0. We have $1 \cdot w = w = \text{Id}(w)$. Following the same argument, we have $(\epsilon \otimes \text{Id}) \circ \Delta(w) = 1 \otimes w$, and $1 \cdot w = w = \text{Id}(w)$. Both diagrams in the definition commute, so W is a coalgebra.

For any $w_1, w_2 \in \mathcal{W}$, we can see that $w_1 \sqcup w_2 = \mathcal{E}$ if and only if $w_1 = w_2 = \mathcal{E}$. Following the same argument as in the concatenation-deshuffle bialgebra, we can see that ϵ is an algebra homomorphism.

For any $k \in K$, we have $\Delta \circ u(k) = k\Delta(\mathcal{E}) = k\mathcal{E} \otimes \mathcal{E} = u_{W \otimes W}(k)$.

For any $w_1, w_2 \in \mathcal{W}$, if w_1 or w_2 is empty (without loss of generality, we can assume that $w_1 = \mathcal{E}$), then $\Delta \circ m(w_1 \otimes w_2) = \Delta(w_2)$. We can see that $\Delta \otimes \Delta(w_1 \otimes w_2) = \mathcal{E} \otimes \mathcal{E} \otimes \Delta(w_2)$ and $m_{13,24}(\mathcal{E} \otimes \mathcal{E} \otimes \Delta(w_2)) = \Delta(w_2) = \Delta \circ m(w_1 \otimes w_2)$. For any $w_1 = a_1 \cdots a_k, w_2 = a_{k+1} \cdots a_{k+l} \in \mathcal{W}$ where $k, l \in \mathbb{Z}_+$, we have

$$\Delta \circ m(w_1 \otimes w_2) = \sum_{i=0}^{k+l} \sum_{\sigma \text{ permutes } \{1, \dots, k+l\}} a_{\sigma(1)} \cdots a_{\sigma(i)} \otimes a_{\sigma(i+1)} \cdots a_{\sigma(k+l)},$$

where $\sigma^{-1}(1) < \cdots < \sigma^{-1}(k)$ and $\sigma^{-1}(k+1) < \cdots < \sigma^{-1}(k+l)$. We have

$$\Delta \otimes \Delta(w_1 \otimes w_2) = \sum_{i=0}^k \sum_{j=0}^l a_1 \cdots a_i \otimes a_{i+1} \cdots a_k \otimes a_{k+1} \cdots a_{k+j} \otimes a_{k+j+1} \cdots a_{k+l}.$$

We can see that every summand in $\Delta \circ m(w_1 \otimes w_2)$ is also in $m_{13,24} \circ (\Delta \otimes \Delta)(w_1 \otimes w_2)$ since we can choose the first and third terms such that their product can be permuted to give the first term in $\Delta \circ m(w_1 \otimes w_2)$. The second term in $\Delta \circ m(w_1 \otimes w_2)$ can be obtained in a similar way. We can see that every summand in $m_{13,24} \circ (\Delta \otimes \Delta)(w_1 \otimes w_2)$ is also in $\Delta \circ m(w_1 \otimes w_2)$ by choosing the appropriate permutation σ . We conclude that Δ is also an algebra homomorphism. Therefore, W constructed on shuffle and deconcatenation is indeed a bialgebra.

2.3 Commutativity and Grading

Next we want to introduce the transposition operation $\tau : A \otimes A \rightarrow A \otimes A$ such that $\tau(a_1 \otimes a_2) = a_2 \otimes a_1$. We can see that $m_{1,3,2,4} = \text{Id} \otimes \tau \otimes \text{Id}$. In addition to moving the factors around so that the correct factors are multiplied together, the transposition operation can also be used to define the commutativity of algebra.

Definition 6. An algebra A is commutative if the following diagram commutes.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & \searrow m & \downarrow m \\ & & A \end{array}$$

Similarly, we can define the cocommutativity of coalgebra.

Definition 7. A coalgebra C is cocommutative if the following diagram commutes.

$$\begin{array}{ccc} C & & \\ \downarrow \Delta & \searrow \Delta & \\ C \otimes C & \xrightarrow{\tau} & C \otimes C \end{array}$$

We know that the concatenation operation is not commutative. For example $m(a \otimes b) = ab \neq ba = m(b \otimes a)$ where $a, b \in \Omega$. Hence, the concatenation-deshuffle bialgebra is not commutative. However, the deshuffle operation is cocommutative. For any non-empty word $a_1 \cdots a_n \in \mathcal{W}$, where $a_1, \dots, a_n \in \Omega, n \in \mathbb{Z}_+$, we have

$$\Delta(a_1 \cdots a_n) = \sum_{I \subset \{1, \dots, n\}} a_I \otimes a_{\{1, \dots, n\} \setminus I}.$$

For any summand $a_I \otimes a_{\{1, \dots, n\} \setminus I}$, let $J = \{1, \dots, n\} \setminus I$, and we can see that $a_J \otimes a_{\{1, \dots, n\} \setminus J} = a_{\{1, \dots, n\} \setminus I} \otimes a_I$ is also a summand. Hence, the concatenation-deshuffle bialgebra is cocommutative.

Similarly, by definition, we can see that the shuffle operation is commutative whereas the deconcatenation operation is not cocommutative. Therefore, the shuffle-deconcatenation bialgebra is commutative but not cocommutative.

Next we introduce the concept of grading of a bialgebra. We need the concept of grading of a vector space first.

Definition 8. *A vector space V over a field K is graded (or $\mathbb{Z}_{\geq 0}$ -graded to be more precise) if it has a direct sum decomposition $V = \bigoplus_{i=0}^{\infty} V_i$, where V_1, V_2, \dots are vector spaces over K . For any $i \in \mathbb{Z}_{\geq 0}$, the vector space V_i is called the graded piece of degree i and the elements of V_i are called homogeneous of degree i .*

We observe that if V is a graded vector space, then $V \otimes V$ is also graded. Specifically, the graded piece of degree n is

$$(V \otimes V)_n = \bigoplus_{j=0}^n V_j \otimes V_{n-j}$$

for any $n \in \mathbb{Z}_{\geq 0}$.

Let \mathcal{W}_n be the set of words over the alphabet Ω with length n where $n \in \mathbb{Z}_{\geq 0}$, and let $W_n = \text{span}_K \mathcal{W}_n$. We have $W = \bigcup_{i=0}^{\infty} \mathcal{W}_i$ and we can see that $W = \bigoplus_{i=0}^{\infty} W_i$, so W is graded with W_i being the graded piece of degree i for any $i \in \mathbb{Z}_{\geq 0}$.

Specifically, we want to define the connected graded vector spaces, which will be useful in proving a theorem coming up later.

Definition 9. *A graded vector space V over K is connected if $V_0 \cong K$.*

Since $W_0 = \text{span}_K \{\mathcal{E}\} \cong K$, we can see that W is a connected graded vector space.

In addition, we can see that any vector space V is trivially graded if we let $V_0 = V$ and $V_i = \{0\}$ for any $i \in \mathbb{Z}_+$. However, this trivial grading is usually not helpful when we consider graded linear maps.

Definition 10. *A linear map $f : V \rightarrow W$, where V and W are graded vector spaces, is graded if $f(V_n) \subset W_n$ for any $n \in \mathbb{Z}_{\geq 0}$.*

With graded vector spaces and graded linear maps, we can define graded algebras, graded coalgebras and graded bialgebras.

Definition 11. *An algebra, coalgebra, or bialgebra is graded if the underlying vector space and all the defining maps are graded.*

As we discussed before, for any $n \in \mathbb{Z}_{\geq 0}$, the graded piece of degree n of $W \otimes W$ is $\bigoplus_{j=0}^n W_j \otimes W_{n-j}$. With our definition of concatenation-deshuffle and shuffle-deconcatenation bialgebras, we can see that both multiplications map the graded piece of degree n of $W \otimes W$ to the graded piece of degree n of W , and both coproducts map the graded piece of degree n of W to the graded piece of degree n of $W \otimes W$. Therefore, both bialgebra are graded.

We want to prove the following results about the graded connected bialgebras, which will be referred to later.

Theorem 3. *Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded connected bialgebra over K .*

1. $u : K \rightarrow A_0$ is an isomorphism.
2. $\epsilon|_{A_0}$ is the inverse to the isomorphism in 1.
3. $\ker \epsilon = \bigoplus_{n=1}^{\infty} A_n$.
4. For any $x \in \ker \epsilon$, $\Delta(x) = \mathcal{E} \otimes x + x \otimes \mathcal{E} + \tilde{\Delta}(x)$ where $\mathcal{E} = u(1)$ and $\tilde{\Delta}(x) \in \ker \epsilon \otimes \ker \epsilon$.

Proof. 1. Since A is connected, we have $A_0 \cong K$, so there exists $u_1 \in A_0$ such that $m(u_1 \otimes x) = m(x \otimes u_1) = x$ for any $x \in A_0$. By the definition of algebra, we have $m(u(1) \otimes x) = m(x \otimes u(1)) = x$ for any $x \in A$. Since $A_0 \subset A$ and the multiplicative identity is unique in A_0 , we have $u_1 = u(1)$. Since $A_0 \cong K$, any element in A_0 can be expressed as ku_1 for some $k \in K$ and we have $u(k) = ku(1) = ku_1$, so $u : K \rightarrow A_0$ is surjective. In addition, the only solution to $0 = ku_1 = ku(1) = u(k)$ is $k = 0$, so the map is injective. Since $u : K \rightarrow A_0$ is an algebra homomorphism, it is an isomorphism.

2. Since ϵ is an algebra homomorphism, in the proof of Theorem 2, we have shown that $\epsilon \circ u = \text{Id}$. Since any element in A_0 can be expressed as $ku(1)$ for some $k \in K$, we have $u \circ \epsilon(ku(1)) = ku \circ (\epsilon \circ u)(1) = ku(1)$, so $u \circ \epsilon = \text{Id}$. Hence, $\epsilon|_{A_0}$ is the inverse to $u : K \rightarrow A_0$.
3. Suppose that for some $x \in \bigoplus_{n=1}^{\infty} A_n$, $\epsilon(x) = k \neq 0$ for some $k \in K$, then $x = ku(1) = ku_1 \in A_0$. Since $x \neq 0$, it contradicts the definition of direct sum. Suppose $\epsilon(ku_1) = 0$ for some $ku_1 \in A_0$ where $k \neq 0$, we have $0 = k\epsilon \circ u(1) = k$, which is a contradiction. Hence, $\ker \epsilon = \bigoplus_{n=1}^{\infty} A_n$.
4. For any $x \in \ker \epsilon$, x cannot be expressed as $k\mathcal{E}$ for some nonzero $k \in K$, so $x \otimes \mathcal{E} \neq \mathcal{E} \otimes x$ by part 3. Since $(\text{Id} \otimes \epsilon) \circ \Delta(x) = x \otimes 1$, there is a $x \otimes \mathcal{E}$ summand in $\Delta(x)$. Similarly, since $(\epsilon \otimes \text{Id}) \circ \Delta(x) = 1 \otimes x$, there is a $\mathcal{E} \otimes x$ summand in $\Delta(x)$. For any other summand $x_1 \otimes x_2$ of

$\Delta(x)$, where $x_1, x_2 \in A$, we have $\text{Id} \circ \epsilon(x_1 \otimes x_2) = 0 = \epsilon \circ \text{Id}(x_1 \otimes x_2)$, so $x_1 \otimes x_2 \in \ker \epsilon \otimes \ker \epsilon$. Since $\ker \epsilon \otimes \ker \epsilon$ is a vector space, we have $\Delta(x) \in \ker \epsilon \otimes \ker \epsilon$. □

For a bialgebra A and $a \in A$, if $\Delta(a) = a \otimes a$, then we say that the element a is group-like. We can easily see that the only group-like element in both concatenation-deshuffle and shuffle-deconcatenation bialgebras is \mathcal{E} . For a bialgebra A and $a \in A$, if $\Delta(a) = \mathcal{E} \otimes a + a \otimes \mathcal{E}$, then we say that the element a is primitive. If $\Delta(a) = \mathcal{E} \otimes a + a \otimes \mathcal{E} + \tilde{\Delta}(a)$, then we call $\mathcal{E} \otimes a + a \otimes \mathcal{E}$ the primitive part.

2.4 Hopf Algebra

Before we define Hopf algebras, we want to introduce the concept of the convolution product.

Definition 12. For any algebra A and coalgebra C , let $f, g : C \rightarrow A$ be linear maps. The convolution product of f and g is

$$f \star g = m \circ (f \otimes g) \circ \Delta.$$

In addition, we need to introduce a linear map called the antipode to define Hopf algebras.

Definition 13. A bialgebra B is a Hopf algebra if there exists a linear map $S : B \rightarrow B$, which we call the antipode, such that the following diagram commutes.

$$\begin{array}{ccccc}
 & & B \otimes B & \xrightarrow{S \otimes \text{Id}} & B \otimes B \\
 & \nearrow \Delta & & & \downarrow m \\
 B & \xrightarrow{\epsilon} & K & \xrightarrow{u} & B \\
 & \searrow \Delta & & & \uparrow m \\
 & & B \otimes B & \xrightarrow{\text{Id} \otimes S} & B \otimes B
 \end{array}$$

By the definition of convolution product, we can see that this diagram shows that the antipode has to satisfy the relation $S \star \text{Id} = \text{Id} \star S = u \circ \epsilon$.

We want to determine whether the concatenation-deshuffle bialgebra and the shuffle-deconcatenation bialgebra are Hopf algebras. Since we have shown before that both bialgebras are graded and connected, we could use the following proposition to help us find the antipode.

Theorem 4. 1. For any Hopf algebra A , the antipode S is an algebra anti-automorphism, i.e., S is an algebra isomorphism, $S(\mathcal{E}) = \mathcal{E}$ and $S(ab) = S(b)S(a)$ for any $a, b \in A$.

2. For any Hopf algebra A , if A is commutative or cocommutative then $S \circ S = \text{Id}$.

3. For any graded connected bialgebra A , A has a unique antipode S which is determined recursively. In addition, S is a graded map, so A is a graded Hopf algebra.

The proof of this theorem can be found in Section 1.4 of [1]. However, since the third statement gives information about the antipode, which is required to prove whether a bialgebra is a Hopf algebra, we will go through the proof of that statement in detail. In particular, the proof gives us the recursive relation with which we can compute the antipode.

Proof. Since A is graded, we have $A = \bigoplus_{n=0}^{\infty} A_n$. Since A is connected, we have $A_0 \cong K$. Since $S(\mathcal{E}) = \mathcal{E} \in A_0$, by the universal property of vector spaces, we can see that $S|_{A_0} = \text{Id}$. For any $x \in A_n$ where $n \in \mathbb{Z}_+$, by Theorem 3, we have $\Delta(x) = \mathcal{E} \otimes x + x \otimes \mathcal{E} + \tilde{\Delta}(x)$, where $\tilde{\Delta}(x) \in \ker \epsilon \otimes \ker \epsilon$. We can write

$$\tilde{\Delta}(x) = \sum_i x_{i,1} \otimes x_{i,2}.$$

By our discussion of the grading of the tensor product of vector spaces before, we know that $x_{i,1}$ and $x_{i,2}$ are homogeneous of degrees strictly less than n and the degrees add up to n .

Since $x \in \ker \epsilon$ by Theorem 3, we have

$$0 = u \circ \epsilon(x) = S * \text{Id}(x) = x + S(x) + \sum_i S(x_{i,1})x_{i,2},$$

so

$$S(x) = -x - \sum_i S(x_{i,1})x_{i,2}.$$

With this recursive relation, we can determine S . Since the base case is $S(x) = -x$, and the degree of $x_{i,1}$ and $x_{i,2}$ add up to n , we can see that S is graded. \square

By induction, we can show that the antipode for both bialgebras is

$$S(w_1 \cdots w_k) = (-1)^k w_k \cdots w_1,$$

for any $w_1 \cdots w_k \in W$ where $k \in \mathbb{Z}_{\geq 0}$.

2.5 Incidence Structure

In order to introduce the incidence structure, we need to first define intervals inside a poset.

Definition 14. Let (P, \leq) be a partially ordered set (poset). An interval in P is a nonempty subposet of the form

$$[x, y] = \{z \in P \mid x \leq z \leq y\},$$

for any $x, y \in P$. We denote the set of all intervals in P as $\text{int}(P)$.

When defining an incidence structure on a poset, we want the poset to be locally finite.

Definition 15. *A poset is locally finite if all its intervals are finite.*

We can introduce the following incidence coalgebra structure on locally finite posets.

Definition 16. *Let P be a locally finite poset and $C = \text{span}_K \text{int}(P)$ be the vector space over K constructed from the basis $\text{int}(P)$. The incidence coalgebra of P is the coalgebra C with the coproduct*

$$\Delta([x, y]) = \sum_{z \in [x, y]} [x, z] \otimes [z, y],$$

and the counit

$$\epsilon([x, y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases},$$

for any $[x, y] \in \text{int}(P)$.

We need to check that C is indeed a coalgebra. Similar to our discussion on the concatenation-deshuffle and shuffle-deconcatenation bialgebras, by the universal property of vector spaces, we need to verify the properties of the coalgebra only on the basis.

For any $[x, y] \in C$, where $x, y \in P$, We can see that

$$(\Delta \otimes \text{Id}) \circ \Delta([x, y]) = \sum_{w, z \in P, x \leq w \leq z \leq y} [x, w] \otimes [w, z] \otimes [z, y] = (\text{Id} \otimes \Delta) \circ \Delta([x, y]),$$

so Δ is co-associative.

For any $[x, y] \in C$, where $x, y \in P$, we have $(\text{Id} \circ \epsilon) \circ \Delta([x, y]) = [x, y] \otimes 1$, since any summand $[x, z] \otimes [z, y]$ is mapped to 0 by $\text{Id} \otimes \epsilon$ if $z \neq y$. We have $1 \cdot [x, y] = [x, y] = \text{Id}([x, y])$. Similarly, we have $(\epsilon \circ \text{Id}) \circ \Delta([x, y]) = 1 \otimes [x, y]$, and $1 \cdot [x, y] = [x, y] = \text{Id}([x, y])$. Therefore, we conclude that C is indeed a coalgebra.

We can also introduce the incidence algebra structure on locally finite posets. However, the definitions will be a bit involved and we will not use the multiplication associated with the incidence algebra. Therefore, we will skip further discussion on the incidence structure.

3 Word Bialgebras and Cointeraction

3.1 Concatenation-Deshuffle Bialgebra on Two Disjoint Alphabets

We can try to develop the concatenation-deshuffle bialgebra and construct a similar structure on two disjoint alphabets.

Let Ω and Ω' be two disjoint alphabets. Let \mathcal{W} be the set of all words over the alphabet $\Omega \cup \Omega'$, and let $W = \text{span}_K \mathcal{W}$ be the vector space of formal linear combinations of words over the field K . By the definition of W , we can see that \mathcal{W} is a basis of W .

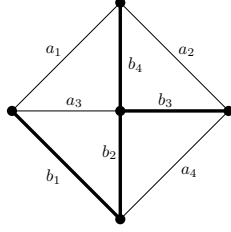
Note that if there were no additional restriction, we could define a new alphabet $\Gamma = \Omega \cup \Omega'$ such that W would become the same concatenation-deshuffle bialgebra described in the previous section. We want to introduce a correspondence relation between the letters from two different alphabets.

Consider the map $f : \Omega \rightarrow \mathcal{P}(\Omega')$, where $\mathcal{P}(\Omega') = \{S \mid S \subset \Omega'\}$ is the power set of Ω' , which maps each letter in Ω to the set of letters it corresponds to in Ω' . Note that for each $a \in \Omega$ in w , the letters of $f(a)$ also appear in w , and we choose particular copies of the letters of $f(a)$ and consider that they are associated with that particular a .

For example, let $\Omega = \{a_1, a_2\}$ and $\Omega' = \{b_1, b_2, b_3\}$. Let $f(a_1) = \{b_1, b_2\}$ and $f(a_2) = \{b_1, b_2, b_3\}$. Without further explanation, for the sake of consistency, a_i will be letters in Ω and b_j will be letters in Ω' for any $i, j \in \mathbb{Z}_+$ for this whole paper. We consider the word $w = a_1 b_1 b_2 a_2 b_1 b_3 \in \mathcal{W}$, where a_1 corresponds to b_1, b_2 of indices 2, 3 and a_2 corresponds to b_1, b_2, b_3 of indices 5, 3, 6 (Note that the same letter in Ω' inside a word can be associated with multiple different letters in Ω inside that word; here b_2 of index 3 is associated with both a_1 of index 1 and a_2 of index 4). The map f is insufficient to determine the exact correspondence relation within each word. We can see that there are two different b_1 's in w . If a_1 corresponds to b_1, b_2 of indices 5, 3 and a_2 corresponds to b_1, b_2, b_3 of indices 2, 3, 6, we have a difference correspondence relation that also satisfies f .

For any $w = w_1 \cdots w_k \in W$ where $k \in \mathbb{Z}_+$, we introduce the map $\tilde{f}_w : \{1, \dots, k\} \rightarrow \mathcal{P}(\{1, \dots, k\})$ to determine uniquely the correspondence relation within the word w . For any $i = 1, \dots, k$, if $w_i \in \Omega$, let $\tilde{f}_w(i)$ be the set of indices of the one copy of elements of $f(w_i)$ that w_i corresponds to. If $w_i \notin \Omega$, let $\tilde{f}_w(i) = \emptyset$. For the word w that we described above, we have $\tilde{f}_w(1) = \{2, 3\}$, $\tilde{f}_w(4) = \{3, 5, 6\}$, and $\tilde{f}_w(2) = \tilde{f}_w(3) = \tilde{f}_w(5) = \tilde{f}_w(6) = \emptyset$.

Although these two new maps make the correspondence uniquely determined within each word, the map f introduces restrictions on the underlying vector space with which we are going to construct the bialgebra structure. For example, with our previous Ω , Ω' and f , we can see that $w = a_1$ is not a valid word anymore, since the associated letters b_1 and b_2 are missing. In addition, we do not want letters in Ω' that are not associated with any letter in Ω , so with our previous Ω , Ω' and f , we want to exclude words such as $w = b_1 b_2$. Hence, if a valid word has only one letter in Ω , all letters in Ω' will be associated with that letter in Ω . In this case there is only one way to define \tilde{f} , and we will omit it in the following discussion. Let \mathcal{W}_f be the set of all valid words over the alphabet $\Omega \cup \Omega'$ with respect to f , and let $W_f = \text{span}_K \mathcal{W}_f$ be the vector space of formal linear combinations of words over the field K . By the definition of W_f , we can see that \mathcal{W}_f is a basis of W_f .



The definition of the bialgebra W_f and the maps f have graph theoretical motivation. The detail of the definition can be found in [4]. We will briefly go over an example to illustrate the relation to graph theory. For the graph G on the left, we fix a spanning tree T . We associate each edge in T with a letter in Ω' and each edge not in T with a letter in Ω . Since T is the spanning tree of G , adding an additional edge e would create a unique cycle C . We call this cycle C the fundamental cycle of the edge e .

For any edge e not in the spanning tree T , we want $\{e\} \cup f(e)$ to be the fundamental cycle of e . Hence, we have $f(a_1) = \{b_1, b_2, b_4\}$, $f(a_2) = \{b_3, b_4\}$, $f(a_3) = \{b_1, b_2\}$, and $f(a_4) = \{b_2, b_3\}$. Every word consisting of $a_1, \dots, a_4, b_1, \dots, b_4$ gives an ordering of the edges in the graph. Note that there are a few key differences between the graph theoretical motivation and our bialgebra. First, letters in Ω can only appear once in this situation, while they can appear multiple times in the words in our bialgebra. Moreover, if we remove the edges a_1 and a_2 from the previous graph, then b_4 becomes an unassociated letter in Ω' , which is excluded in our definition.

Similar to the concatenation on one alphabet, it suffices to define the multiplication map \hat{m} over the basis $\mathcal{W}_f \otimes \mathcal{W}_f$. Let \hat{m} be the concatenation of words. We can easily see that the concatenation of two words in \mathcal{W}_f is still in \mathcal{W}_f with the unique correspondence relation preserved in both subwords. Let i be the inclusion map. By the universal property of vector spaces, there exists a unique linear map m which we define as the multiplication for the bialgebra. We call this multiplication concatenation.

$$\begin{array}{ccc} \mathcal{W}_f \otimes \mathcal{W}_f & \xrightarrow{i} & \mathcal{W}_f \otimes \mathcal{W}_f \\ & \searrow \hat{m} & \downarrow m \\ & & \mathcal{W}_f \end{array}$$

For example, let $w_1 = a_1 b_1 b_2 a_2 b_1 b_3$ be the word described above and $w_2 = a_1 b_1 b_2 a_2 b_3$ where $\tilde{f}_{w_2}(1) = \{2, 3\}$, $\tilde{f}_{w_2}(4) = \{2, 5\}$, and $\tilde{f}_{w_2}(2) = \tilde{f}_{w_2}(3) = \tilde{f}_{w_2}(5) = \emptyset$. We have $w_3 = m(w_1 \otimes w_2) = a_1 b_1 b_2 a_2 b_1 b_3 a_1 b_1 b_2 a_2 b_3$ with $\tilde{f}_{w_3}(1) = \{2, 3\}$, $\tilde{f}_{w_3}(4) = \{3, 5, 6\}$, $\tilde{f}_{w_3}(7) = \{8, 9\}$, $\tilde{f}_{w_3}(10) = \{8, 11\}$, and $\tilde{f}_{w_3}(2) = \tilde{f}_{w_3}(3) = \tilde{f}_{w_3}(5) = \tilde{f}_{w_3}(6) = \tilde{f}_{w_3}(8) = \tilde{f}_{w_3}(9) = \tilde{f}_{w_3}(11) = \emptyset$.

Similarly, by the universal property of vector spaces, we can define the coproduct $\hat{\Delta}$ on the basis \mathcal{W}_f to obtain the unique coproduct map $\Delta : \mathcal{W}_f \rightarrow \mathcal{W}_f \otimes \mathcal{W}_f$.

$$\begin{array}{ccc} \mathcal{W}_f & \xrightarrow{i} & \mathcal{W}_f \\ & \searrow \hat{\Delta} & \downarrow \Delta \\ & & \mathcal{W}_f \otimes \mathcal{W}_f \end{array}$$

It is difficult to write out an exact expression for this coproduct, but the idea is easy to grasp. The coproduct Δ deshuffles the word with respect to

the alphabet Ω and puts letters in the alphabet Ω' in the result in a way such that the correspondence relation and the relative position are preserved. For example, we have

$$\Delta(w_1) = a_1 b_1 b_2 a_2 b_1 b_3 \otimes \mathcal{E} + \mathcal{E} \otimes a_1 b_1 b_2 a_2 b_1 b_3 + a_1 b_1 b_2 \otimes b_2 a_2 b_1 b_3 + b_2 a_2 b_1 b_3 \otimes a_1 b_1 b_2$$

and

$$\Delta(w_2) = a_1 b_1 b_2 a_2 b_3 \otimes \mathcal{E} + \mathcal{E} \otimes a_1 b_1 b_2 a_2 b_3 + a_1 b_1 b_2 \otimes b_1 a_2 b_3 + b_1 a_2 b_3 \otimes a_1 b_1 b_2.$$

This definition of coproduct explains why we want to exclude unassociated letters in Ω' . If we concatenate w_1 with unassociated $b_1 b_2$, then the question about where to put unassociated $b_1 b_2$ in the coproduct of this word will become problematic.

We have $\Delta(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}$, and for any $w \in \mathcal{W}_f$ in which there are n letters in Ω for any $n \in \mathbb{Z}_+$, we have

$$\Delta(w) = \sum_{I \subset \{1, \dots, n\}} w_I^* \otimes w_{\{1, \dots, n\} \setminus I}^*,$$

where w_I^* is the word formed from w by choosing all i -th letters in Ω for $i \in I$ and filling in letters from Ω' in a way such that the correspondence relation and the relative position are preserved.

We use the same definitions of unit and counit in the concatenation-deshuffle bialgebra on one alphabet.

We need to verify that our W_f constructed in the way described above is indeed a bialgebra. By the universal property of vector spaces, we only need to verify that the diagrams in the definition of bialgebras commute over the basis of W_f .

For any $w_1, w_2, w_3 \in \mathcal{W}_f$, we can see that m is associative since

$$m(m(w_1 \otimes w_2) \otimes w_3) = \overline{w_1 w_2 w_3} = m(w_1 \otimes m(w_2 \otimes w_3)).$$

For any $w \in \mathcal{W}_f$, we have $\text{Id} \otimes u(w \otimes 1) = w \otimes \mathcal{E}$, so $m(w \otimes \mathcal{E}) = w = \text{Id}(w)$. Similarly, we have $u \otimes \text{Id}(1 \otimes w) = \mathcal{E} \otimes w$, so $m(\mathcal{E} \otimes w) = w = \text{Id}(w)$. Both diagrams in the definition commute, so W_f is an algebra.

If $w = \mathcal{E}$, we have $(\Delta \circ \text{Id}) \circ \Delta(w) = \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E} = (\text{Id} \circ \Delta) \circ \Delta(w)$. For any $w \in \mathcal{W}_f$ in which there are n letters in Ω , where $n \in \mathbb{Z}_+$, we have

$$(\Delta \circ \text{Id}) \circ \Delta(w) = \sum_{I, J, K \text{ disjoint}, I \cup J \cup K = \{1, \dots, n\}} w_I^* \otimes w_J^* \otimes w_K^* = (\text{Id} \circ \Delta) \circ \Delta(w),$$

so Δ is co-associative.

For any $w \in \mathcal{W}_f$, we have $(\text{Id} \otimes \epsilon) \circ \Delta(w) = w \otimes 1$, since all summands other than $w \otimes \mathcal{E}$ are mapped to 0 because ϵ maps all non-empty words to 0. We have $1 \cdot w = w = \text{Id}(w)$. Following the same argument, we have $(\epsilon \otimes \text{Id}) \circ \Delta(w) = 1 \otimes w$, and $1 \cdot w = w = \text{Id}(w)$. Both diagrams in the definition commute, so W is a coalgebra.

For any $w \in \mathcal{W}_f$, we have $(\text{Id} \otimes \epsilon) \circ \Delta(w) = w \otimes 1$, since all summands other than $w \otimes \mathcal{E}$ are mapped to 0 because ϵ maps all non-empty words to 0. We have $1 \cdot w = w = \text{Id}(w)$. Following the same argument, we have $(\epsilon \otimes \text{Id}) \circ \Delta(w) = 1 \otimes w$, and $1 \cdot w = w = \text{Id}(w)$. Both diagrams in the definition commute, so W is a coalgebra.

For any $k \in K$, we have $u(k) = k\mathcal{E}$, so $\epsilon \circ u(k) = k\epsilon(\mathcal{E}) = k = \text{Id}(k)$.

For any $w_1, w_2 \in \mathcal{W}_f$, if w_1 or w_2 is empty (without loss of generality, we can assume that $w_1 = \mathcal{E}$), then

$$\epsilon \circ m(w_1 \otimes w_2) = \epsilon(w_2) = 1 \cdot \epsilon(w_2) = \epsilon(w_1)\epsilon(w_2) = m_K \circ (\epsilon \otimes \epsilon)(w_1 \otimes w_2).$$

If w_1 and w_2 are non-empty, then $m(w_1 \otimes w_2)$ is also non-empty, so

$$\epsilon \circ m(w_1 \otimes w_2) = 0 = 0 \cdot 0 = \epsilon(w_1)\epsilon(w_2) = m_K \circ (\epsilon \otimes \epsilon)(w_1 \otimes w_2).$$

We conclude that the counit ϵ is an algebra homomorphism.

For any $k \in K$, we have $\Delta \circ u(k) = k\Delta(\mathcal{E}) = k\mathcal{E} \otimes \mathcal{E} = u_{W_f \otimes W_f}(k)$.

For any $w_1, w_2 \in \mathcal{W}_f$, if w_1 or w_2 is empty (without loss of generality, we can assume that $w_1 = \mathcal{E}$), then $\Delta \circ m(w_1 \otimes w_2) = \Delta(w_2)$. We can see that $\Delta \otimes \Delta(w_1 \otimes w_2) = \mathcal{E} \otimes \mathcal{E} \otimes \Delta(w_2)$ and $m_{13,24}(\mathcal{E} \otimes \mathcal{E} \otimes \Delta(w_2)) = \Delta(w_2) = \Delta \circ m(w_1 \otimes w_2)$. For any $w_1, w_2 \in \mathcal{W}_f$ where there are k letters in Ω in w_1 and l letters in Ω' in w_2 , $k, l \in \mathbb{Z}_+$, we have

$$\Delta \circ m(w_1 \otimes w_2) = \sum_{I \in \{1, \dots, k+l\}} \overline{w_1 w_{2I}^*} \otimes \overline{w_1 w_{2\{1, \dots, k+l\} \setminus I}^*},$$

and

$$\begin{aligned} & m_{13,24} \circ (\Delta \otimes \Delta)(w_1 \otimes w_2) \\ &= m_{13,24} \left(\sum_{I \in \{1, \dots, k\}} \sum_{J \in \{k+1, \dots, k+l\}} w_{1I}^* \otimes w_{1\{1, \dots, k\} \setminus I}^* \otimes w_{2J}^* \otimes w_{2\{k+1, \dots, k+l\} \setminus J}^* \right) \\ &= \sum_{I \in \{1, \dots, k\}, J \in \{k+1, \dots, k+l\}} \overline{w_1 w_{2I \cup J}^*} \otimes \overline{w_1 w_{2\{1, \dots, k+l\} \setminus (I \cup J)}^*} \\ &= \Delta \circ m(w_1 \otimes w_2). \end{aligned}$$

We conclude that the coproduct Δ is an algebra homomorphism. Therefore, W constructed on concatenation and deshuffle is indeed a bialgebra.

We can see that the proof is very similar to the concatenation-deshuffle bialgebra on one algebra. This similarity is understandable since in our definition, all letters in Ω' are associated with letters in Ω . In the definition of coproduct, we deshuffle with respect to Ω and bring letters in Ω' accordingly for the ride.

We could also try to define the shuffle-deconcatenation bialgebra on Ω and Ω' that are correlated with the map f . However, we immediately run into the similar problem about unassociated letters in Ω' that we tried to exclude in the definition. For example, let $w_1 = b_1 a_1 b_2 a_2$, where $\tilde{f}_{w_1}(2) = \{1, 3\}$, $\tilde{f}_{w_1}(4) = \{3\}$, and $\tilde{f}_{w_1}(1) = \tilde{f}_{w_1}(3) = \emptyset$, and $w_2 = a_3 b_3$. If we try to define $w_1 \sqcup w_2$ as the

shuffle product with respect to letters in Ω and with letters in Ω' filled in to preserve the correspondence relation and the relative position, then there is no way of filling in b_1, b_2, b_3 into $a_1 a_3 a_2$ that would make sense. Therefore, the shuffle-deconcatenation bialgebra cannot be extended to two alphabets in the same way as the concatenation-deshuffle bialgebra.

Next we can check the commutativity of W_f . Let $w_1 = a_1 b_1$ and $w_2 = a_2 b_2$. We have $m(w_1 \otimes w_2) = a_1 b_1 a_2 b_2 \neq a_2 b_2 a_1 b_1 = m(w_2 \otimes w_1)$, so W_f is not commutative. For any non-empty word w with n letters in Ω , where $n \in \mathbb{Z}_+$, we have

$$\Delta(w) = \sum_{I \subset \{1, \dots, n\}} w_I^* \otimes w_{\{1, \dots, n\} \setminus I}^*$$

For any summand $w_I^* \otimes w_{\{1, \dots, n\} \setminus I}^*$, let $J = \{1, \dots, n\} \setminus I$, and we can see that $w_J^* \otimes w_{\{1, \dots, n\} \setminus J}^* = w_{\{1, \dots, n\} \setminus I}^* \otimes w_I^*$ is also a summand. Hence, W_f is cocommutative.

Furthermore, we want to check whether W_f is a Hopf algebra. We could check whether the antipode $S(w) = (-1)^k w^R$ for any $w \in W_f$ would work, where k is the length of w and W^R is the reverse of the word w . Let $w = a_1 b_1 a_2$, where $\tilde{f}_w(1) = \tilde{f}_w(3) = \{2\}$ and $\tilde{f}_w(2) = \emptyset$. We have

$$\Delta(w) = a_1 b_1 a_2 \otimes \mathcal{E} + \mathcal{E} \otimes a_1 b_1 a_2 + a_1 b_1 \otimes b_1 a_2 + b_1 a_2 \otimes a_1 b_1,$$

so

$$(S \otimes \text{Id}) \circ \Delta(w) = -a_2 b_1 a_1 \otimes \mathcal{E} + \mathcal{E} \otimes a_1 b_1 a_2 + b_1 a_1 \otimes b_1 a_2 + a_2 b_1 \otimes a_1 b_1,$$

and thus

$$S \star \text{Id}(w) = -a_2 b_1 a_1 + a_1 b_1 a_2 + b_1 a_1 b_1 a_2 + a_2 b_1 a_1 b_1.$$

We also have $u \circ \epsilon(w) = u(\epsilon(w)) = 0 \neq (S \otimes \text{Id}) \circ \Delta(w)$, so S is not an antipode of W_f . The main reason why this definition fails is that the concatenation product of each summand can be of different length when a letter in Ω' is associated with multiple letters in Ω .

We could check the grading of W_f first. By Theorem 4, any graded connected bialgebra is a graded Hopf algebra and has a unique antipode that can be determined recursively.

Unlike the concatenation-deshuffle bialgebra on one alphabet, W_f is not graded on the length of words. If we were to grade the bialgebra on the length of words, $w = a_1 b_1 a_2$ described above would be homogeneous of degree 3. However, the coproduct $\Delta(w) = a_1 b_1 a_2 \otimes \mathcal{E} + \mathcal{E} \otimes a_1 b_1 a_2 + a_1 b_1 \otimes b_1 a_2 + b_1 a_2 \otimes a_1 b_1$ would be homogeneous of degree 4 because of the summands $a_1 b_1 \otimes b_1 a_2$ and $b_1 a_2 \otimes a_1 b_1$, so the coproduct map would not be graded. As we have discussed before, this situation happens when a letter in Ω' is associated with multiple letters in Ω .

Since the coproduct is based on deshuffling with respect to Ω , one natural way to fix this issue is to try to grade the bialgebra on the number of letters in Ω . Let $\mathcal{W}_{f,n}$ be the set of words in \mathcal{W}_f with n letters in Ω , and let $W_{f,n} =$

$\text{span}_K \mathcal{W}_{f,n}$. We can see that $W_f = \bigoplus_{n=0}^{\infty} W_{f,n}$, so the vector space W_f is graded. Specifically, we can see that $\mathcal{W}_{f,0} = \{\mathcal{E}\}$, so $W_{f,0} \cong K$.

The graded piece of degree n for $W_f \otimes W_f$ is $\bigoplus_{j=0}^n W_{f,j} \otimes W_{f,n-j}$ for any $n \in \mathbb{Z}_{\geq 0}$. For any $w_1 \otimes w_2 \in W_{f,i} \otimes W_{f,j}$ of degree $(i+j)$, where $i, j \in \mathbb{Z}_{\geq 0}$, w_1 has i letters in Ω and w_2 has j letters in Ω , so the concatenation product $m(w_1 \otimes w_2)$ has $(i+j)$ letters in Ω and is of degree $(i+j)$. Hence, the concatenation product map is graded. We can see that $\Delta(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}$ where both \mathcal{E} and $\mathcal{E} \otimes \mathcal{E}$ are of degree 0. For any $w \in W_{f,n}$ of degree n , where $n \in \mathbb{Z}_+$, we have

$$\Delta(w) = \sum_{I \subset \{1, \dots, n\}} w_I^* \otimes w_{\{1, \dots, n\} \setminus I}^*.$$

By definition w_I^* is of degree $|I|$ and $w_{\{1, \dots, n\} \setminus I}^*$ is of degree $(n - |I|)$, so any summand $w_I^* \otimes w_{\{1, \dots, n\} \setminus I}^*$ is of degree n (here $|I|$ is the cardinality of I). Hence, the coproduct is graded. We conclude that W_f is a graded connected bialgebra. By Theorem 4, W_f is a graded Hopf algebra.

For any $w \in W_f$ with n letters in Ω where $n \in \mathbb{Z}_+$, we have $\epsilon(w) = 0$, so $w \in \ker \epsilon$. By the proof of Theorem 4, we have

$$S(w) = -w - \sum_i S(w_{i,1})w_{i,2}$$

where $\tilde{\Delta}(w) = \sum_i w_{i,1} \otimes w_{i,2}$ is the non-primitive part of $\Delta(w)$.

Consider the example $w = a_1 b_1 a_2$, where $\tilde{f}_w(1) = \tilde{f}_w(3) = \{2\}$ and $\tilde{f}_w(2) = \emptyset$. We have

$$S(w) = -w - S(a_1 b_1) b_1 a_2 - S(b_1 a_2) a_1 b_1 = -a_1 b_1 a_2 + a_1 b_1 b_1 a_2 + b_1 a_2 a_1 b_1.$$

As we can see, $S(w)$ has no clear relation to w , so there is no clean expression for antipode for W_f .

3.2 Incidence Structure on the Concatenation-Deshuffle Bialgebra on Two Disjoint Alphabets

For any word $w \in \mathcal{W}$, let I_w be the set of indices of letters in Ω' . We can see that $\mathcal{P}(I_w)$ is a poset. Since w is of finite length, I_w is also finite, so any interval in $\mathcal{P}(I_w)$ and thus $\mathcal{P}(I_w)$ is locally finite.

Let

$$WI = \bigoplus_{w \in \mathcal{W}_f} \text{span}_K \{(w, [A, B]) \mid [A, B] \in \text{int}(\mathcal{P}(I_w))\},$$

and $C_w = \text{span}_K \text{int}(\mathcal{P}(I_w))$, and we can see that $WI \cong \bigoplus_{w \in \mathcal{W}_f} C_w$. We can see that WI is a vector space over K with the basis

$$\mathcal{B} = \bigcup_{w \in \mathcal{W}_f} \{(w, [A, B]) \mid [A, B] \in \text{int}(\mathcal{P}(I_w))\}.$$

By our discussion in Section 2.5, we can introduce an incidence coalgebra structure on WI with the coproduct

$$\delta(w, [A, B]) = \sum_{A \subset C \subset B} (w, [A, C]) \otimes (w, [C, B]),$$

and the counit

$$\epsilon_\delta(w, [A, B]) = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases},$$

for any $(w, [A, B]) \in \mathcal{B}$ (we only gave the definition on the basis, but both the coproduct and counit can be extended by the universal property of vector spaces).

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{i} & WI \\ & \searrow \delta & \downarrow \delta \\ & & WI \otimes WI \end{array}$$

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{i} & WI \\ & \searrow \hat{\epsilon}_\delta & \downarrow \epsilon_\delta \\ & & K \end{array}$$

This incidence structure also has graph theoretical motivation. Let us consider the relation between W_f and the graphs described in 3.1. We are attaching an interval $[A, B]$ to a word w . Note that $[A, B]$ is an interval in the power set of the edges of the spanning tree T . The edges in A are to be contracted and the edges in $E(T) \setminus B$ are to be removed from the spanning tree. We can get a spanning forest from these changes. This spanning forest induces a cut of the graph, since with any edge between different components of the forest having been cut, the resulting components have the different trees of the spanning forest as their spanning trees.

Instead of using the multiplication map from the incidence algebra structure, we want to define the multiplication map in the following way.

$$m((w_1, [A, B]) \otimes (w_2, [C, D])) = (\overline{w_1 w_2}, [A \cup C^*, B \cup D^*]),$$

for any $w_1, w_2 \in \mathcal{W}_f$, $[A, B] \in \text{int}(\mathcal{P}(I_{w_1}))$, and $[C, D] \in \text{int}(\mathcal{P}(I_{w_2}))$, where $C^* = C + |w_1|$, $D^* = D + |w_1|$, and $|w_1|$ is the length of w_1 .

For example, let $w_1 = a_1 b_1 b_2$ and $w_2 = b_1 a_2 b_3$, and we have

$$m((w_1, [\{3\}, \{2, 3\}]) \otimes (w_2, [\{1\}, \{1, 3\}])) = (a_1 b_1 b_2 b_1 a_2 b_3, [\{3, 4\}, \{2, 3, 4, 6\}]).$$

We want to make sure that $m : V \otimes V \rightarrow V$ has the property that the

underlying vector space V is the same as WI . We can see that the basis of V is

$$\begin{aligned} & \bigcup_{w \in \mathcal{W}_f} \bigcup_{[A, B] \in \text{int}(\mathcal{P}(I_w))} \{(w, [A, B])\} \\ &= \bigcup_{w \in \mathcal{W}_f} \{(w, [A, B]) \mid [A, B] \in \text{int}(\mathcal{P}(I_w))\} = \mathcal{B}, \end{aligned}$$

so $V = WI$.

We can also introduce another coproduct Δ based on the deshuffle coproduct of \mathcal{W}_f . For any $w \in \mathcal{W}_f$ and $[A, B] \in \text{int}(\mathcal{P}(I_w))$, we define the coproduct on \mathcal{B}

$$\hat{\Delta}(w, [A, B]) = \sum_i (w_{i,1}, [A_{w_{i,1}}, B_{w_{i,1}}]) \otimes (w_{i,2}, [A_{w_{i,2}}, B_{w_{i,2}}]),$$

where $\Delta(w) = \sum_i w_{i,1} \otimes w_{i,2}$ for the coproduct on the concatenation-deshuffle bialgebra on two alphabets, and $A_{w_{i,1}}$ (similarly for $B_{w_{i,1}}$, $A_{w_{i,2}}$ and $B_{w_{i,2}}$) is A restricted to letters in Ω' in $w_{i,1}$ with indices adjusted accordingly. By the universal property of vector spaces, there exists a unique linear map on WI which we define as the coproduct Δ .

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{i} & WI \\ & \searrow \hat{\Delta} & \downarrow \Delta \\ & & WI \otimes WI \end{array}$$

For example, let $w = a_1 b_1 b_2 a_2 b_3 \in \mathcal{W}_f$ where $\tilde{f}_{w_2}(1) = \{2, 3\}$, $\tilde{f}_{w_2}(4) = \{2, 5\}$, and $\tilde{f}_{w_2}(2) = \tilde{f}_{w_2}(3) = \tilde{f}_{w_2}(5) = \emptyset$. We have

$$\begin{aligned} \Delta(w, [\{3\}, \{2, 3\}]) &= (a_1 b_1 b_2 a_2 b_3, [\{3\}, \{2, 3\}]) \otimes (\mathcal{E}, [\emptyset, \emptyset]) \\ &\quad + (\mathcal{E}, [\emptyset, \emptyset]) \otimes (a_1 b_1 b_2 a_2 b_3, [\{3\}, \{2, 3\}]) \\ &\quad + (a_1 b_1 b_2, [\{3\}, \{2, 3\}]) \otimes (b_1 a_2 b_3, [\emptyset, \{1\}]) \\ &\quad + (b_1 a_2 b_3, [\emptyset, \{1\}]) \otimes (a_1 b_1 b_2, [\{3\}, \{2, 3\}]). \end{aligned}$$

For any $(w, [A, B]) \in \mathcal{B}$, we define the counit on $\hat{\epsilon}_\Delta(w, [A, B]) = 1$ if $w = \mathcal{E}$ and $\hat{\epsilon}_\Delta(w, [A, B]) = 0$ if w is non-empty. By the universal property of vector spaces, there exists a unique counit map ϵ_Δ .

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{i} & WI \\ & \searrow \hat{\epsilon}_\Delta & \downarrow \epsilon_\Delta \\ & & K \end{array}$$

We define the unit map u for both (WI, m, δ) and (WI, m, Δ) such that $u(k) = k(\mathcal{E}, [\emptyset, \emptyset])$.

We want to verify that both (WI, m, δ) and (WI, m, Δ) are bialgebras. By the universal property of vector spaces, we only need to verify that the diagrams in the definition of bialgebras commute over the basis \mathcal{B} .

For any $(w_1, [A_1, A_2]), (w_2, [B_1, B_2]), (w_3, [C_1, C_2]) \in \mathcal{B}$, we have

$$\begin{aligned} & m(m((w_1, [A_1, A_2]) \otimes (w_2, [B_1, B_2])) \otimes (w_3, [C_1, C_2])) \\ &= (\overline{w_1 w_2 w_3}, [(A_1 \cup B_1^*) \cup C_1^*, (A_2 \cup B_2^*) \cup C_2^*]), \end{aligned}$$

and

$$\begin{aligned} & m((w_1, [A_1, A_2]) \otimes m((w_2, [B_1, B_2]) \otimes (w_3, [C_1, C_2]))) \\ &= (\overline{w_1 w_2 w_3}, [A_1 \cup (B_1 \cup C_1^*)^*, A_2 \cup (B_2 \cup C_2^*)^*]). \end{aligned}$$

We can see that $(A_1 \cup B_1^*) \cup C_1^* = A_1 \cup (B_1 \cup C_1^*)^*$ (likewise $(A_2 \cup B_2^*) \cup C_2^* = A_2 \cup (B_2 \cup C_2^*)^*$), since all indices in B_1 are shifted to the right by $|w_1|$ and all indices in C_1 are shifted to the right by $|w_1| + |w_2|$, so m is associative.

For any $(w, [A, B]) \in \mathcal{B}$, we have

$$m \circ (\text{Id} \otimes u)((w, [A, B]) \otimes 1) = m((w, [A, B]) \otimes (\mathcal{E}, [\emptyset, \emptyset])) = (w, [A, B]),$$

and similarly,

$$m \circ (u \text{ Id})((w, [A, B]) \otimes 1) = m((\mathcal{E}, [\emptyset, \emptyset]) \otimes (w, [A, B])) = (w, [A, B]).$$

We conclude that (WI, m, δ) and (WI, m, Δ) are both algebras.

For any $(w, [A, B]) \in \mathcal{B}$, We can see that

$$\begin{aligned} (\delta \otimes \text{Id}) \circ \delta(w, [A, B]) &= \sum_{A \subset C \subset D \subset B} (w, [A, C]) \otimes (w, [C, D]) \otimes (w, [D, B]) \\ &= (\text{Id} \otimes \delta) \circ \delta(w, [A, B]), \end{aligned}$$

so δ is co-associative.

For any $(w, [A, B]) \in \mathcal{B}$, we have $(\text{Id} \circ \epsilon_\delta) \circ \delta((w, [A, B])) = (w, [A, B]) \otimes 1$, since any summand $(w, [A, C]) \otimes (w, [C, B])$ is mapped to 0 by $\text{Id} \circ \epsilon_\delta$ if $C \neq B$. We have $1 \cdot (w, [A, B]) = (w, [A, B]) = \text{Id}(w, [A, B])$. Similarly, we have $(\epsilon_\delta \circ \text{Id}) \circ \delta(w, [A, B]) = 1 \otimes (w, [A, B])$, and $1 \cdot (w, [A, B]) = (w, [A, B]) = \text{Id}(w, [A, B])$. Therefore, we conclude that (WI, m, δ) is indeed a coalgebra.

For any $(w, [A, B]) \in \mathcal{B}$, We can see that

$$\begin{aligned} & (\Delta \otimes \text{Id}) \circ \Delta(w, [A, B]) \\ &= \sum_i (w_{i,1}, [A_{w_{i,1}}, B_{w_{i,1}}]) \otimes (w_{i,2}, [A_{w_{i,2}}, B_{w_{i,2}}]) \otimes (w_{i,3}, [A_{w_{i,3}}, B_{w_{i,3}}]) \\ &= (\text{Id} \otimes \Delta) \circ \Delta(w, [A, B]), \end{aligned}$$

where $(\Delta \otimes \text{Id}) \circ \Delta(w) = \sum_i w_{i,1} \otimes w_{i,2} \otimes w_{i,3}$, so Δ is co-associative.

For any $(w, [A, B]) \in \mathcal{B}$, we have $(\text{Id} \circ \epsilon_\Delta) \circ \Delta((w, [A, B])) = (w, [A, B]) \otimes 1$, since any summand $(w_{i,1}, [A_{w_{i,1}}, B_{w_{i,1}}]) \otimes (w_{i,2}, [A_{w_{i,2}}, B_{w_{i,2}}])$ is mapped to 0 by $\text{Id} \circ \epsilon_\Delta$ if $w_{i,2}$ is non-empty. We have $1 \cdot (w, [A, B]) = (w, [A, B]) = \text{Id}(w, [A, B])$. Similarly, we have $(\epsilon_\Delta \circ \text{Id}) \circ \delta(w, [A, B]) = 1 \otimes (w, [A, B])$, and $1 \cdot (w, [A, B]) =$

$(w, [A, B]) = \text{Id}(w, [A, B])$. Therefore, we conclude that (WI, m, Δ) is indeed a coalgebra.

For any $k \in K$, we have $u(k) = k(\mathcal{E}, [\emptyset, \emptyset])$, so $\epsilon_\delta \circ u(k) = k\epsilon_\delta(\mathcal{E}, [\emptyset, \emptyset]) = k = \text{Id}(k)$ and $\epsilon_\Delta \circ u(k) = k\epsilon_\Delta(\mathcal{E}, [\emptyset, \emptyset]) = k = \text{Id}(k)$.

For any $(w_1, [A, B]), (w_2, [C, D]) \in \mathcal{B}$, if $A = B$ or $C = D$ (without loss of generality, we can assume that $A = B$), then $\epsilon_\delta(w_1 w_2, [A \cup C^*, B \cup D^*]) = \epsilon_\delta(w_2, [C, D])$, so

$$\begin{aligned} & \epsilon_\delta \circ m((w_1, [A, B]) \otimes (w_2, [C, D])) = \epsilon_\delta(w_1 w_2, [A \cup C^*, B \cup D^*]) \\ & = \epsilon_\delta(w_2, [C, D]) = 1 \cdot \epsilon_\delta(w_2, [C, D]) = \epsilon_\delta(w_1, [A, B])\epsilon_\delta(w_2, [C, D]) \\ & = m_K \circ (\epsilon_\Delta \otimes \epsilon_\Delta)((w_1, [A, B]) \otimes (w_2, [C, D])). \end{aligned}$$

If $A \neq B$ and $C \neq D$, then $A \cup C^* \neq B \cup D^*$, so

$$\begin{aligned} & \epsilon_\delta \circ m((w_1, [A, B]) \otimes (w_2, [C, D])) = 0 = 0 \cdot 0 = \epsilon_\delta(w_1, [A, B])\epsilon_\delta(w_2, [C, D]) \\ & = m_K \circ (\epsilon_\delta \otimes \epsilon_\delta)((w_1, [A, B]) \otimes (w_2, [C, D])). \end{aligned}$$

We conclude that counit ϵ_δ is an algebra homomorphism.

For any $(w_1, [A, B]), (w_2, [C, D]) \in \mathcal{B}$, if w_1 or w_2 is empty (without loss of generality, we can assume that $w_1 = \mathcal{E}$), then

$$\begin{aligned} & \epsilon_\Delta \circ m((w_1, [A, B]) \otimes (w_2, [C, D])) = \epsilon_\Delta(w_2, [C, D]) = 1 \cdot \epsilon_\Delta(w_2, [C, D]) \\ & = \epsilon_\Delta(w_1, [A, B])\epsilon_\Delta(w_2, [C, D]) = m_K \circ (\epsilon_\Delta \otimes \epsilon_\Delta)((w_1, [A, B]) \otimes (w_2, [C, D])). \end{aligned}$$

If w_1 and w_2 are non-empty, then $\overline{w_1 w_2}$ is also non-empty, so

$$\begin{aligned} & \epsilon_\Delta \circ m((w_1, [A, B]) \otimes (w_2, [C, D])) = 0 = 0 \cdot 0 = \epsilon_\Delta(w_1, [A, B])\epsilon_\Delta(w_2, [C, D]) \\ & = m_K \circ (\epsilon_\Delta \otimes \epsilon_\Delta)((w_1, [A, B]) \otimes (w_2, [C, D])). \end{aligned}$$

We conclude that the counit ϵ_Δ is an algebra homomorphism.

For any $k \in K$, we have

$$\delta \circ u(k) = k\delta(\mathcal{E}, [\emptyset, \emptyset]) = k(\mathcal{E}, [\emptyset, \emptyset]) \otimes (\mathcal{E}, [\emptyset, \emptyset]) = u_{WI \otimes WI}(k).$$

For any $(w_1, [A, B]), (w_2, [C, D]) \in \mathcal{B}$, we have

$$\begin{aligned} & \delta \circ m((w_1, [A, B]) \otimes (w_2, [C, D])) \\ & = \sum_{A \cup C^* \subset E \subset B \cup D^*} (\overline{w_1 w_2}, [A \cup C^*, E]) \otimes (\overline{w_1 w_2}, [E, B \cup D^*]), \end{aligned}$$

and

$$\begin{aligned}
& m_{13,24} \circ (\delta \otimes \delta)((w_1, [A, B]) \otimes (w_2, [C, D])) \\
&= m_{13,24} \left(\sum_{\substack{A \subset F \subset B \\ C \subset G \subset D}} (w_1, [A, F]) \otimes (w_1, [F, B]) \otimes (w_2, [C, G]) \otimes (w_2, [G, D]) \right) \\
&= \sum_{\substack{A \subset F \subset B \\ C \subset G \subset D}} (\overline{w_1 w_2}, [A \cup C^*, F \cup G^*]) \otimes (\overline{w_1 w_2}, [F \cup G^*, B \cup D^*]) \\
&= \sum_{A \cup C^* \subset E \subset B \cup D^*} (\overline{w_1 w_2}, [A \cup C^*, E]) \otimes (\overline{w_1 w_2}, [E, B \cup D^*]) \\
&= \delta \circ m((w_1, [A, B]) \otimes (w_2, [C, D])),
\end{aligned}$$

since $A \cup C^* \subset F \cup G^* \subset B \cup D^*$, and we can let $E = F \cup G^*$. We conclude that the coproduct δ is an algebra homomorphism. Therefore, (WI, m, δ) is a bialgebra.

For any $k \in K$, we have

$$\Delta \circ u(k) = k\Delta(\mathcal{E}, [\emptyset, \emptyset]) = k(\mathcal{E}, [\emptyset, \emptyset]) \otimes (\mathcal{E}, [\emptyset, \emptyset]) = u_{WI \otimes WI}(k).$$

For any $(w_1, [A, B]), (w_2, [C, D]) \in \mathcal{B}$, we have

$$\begin{aligned}
& \Delta \circ m((w_1, [A, B]) \otimes (w_2, [C, D])) \\
&= \sum_i (\overline{w_1 w_2}_{i,1}, [(A \cup C^*)_{\overline{w_1 w_2}_{i,1}}, (B \cup D^*)_{\overline{w_1 w_2}_{i,1}}]) \\
&\quad \otimes (\overline{w_1 w_2}_{i,2}, [(A \cup C^*)_{\overline{w_1 w_2}_{i,2}}, (B \cup D^*)_{\overline{w_1 w_2}_{i,2}}]),
\end{aligned}$$

where $\Delta(\overline{w_1 w_2}) = \sum_i \overline{w_1 w_2}_{i,1} \otimes \overline{w_1 w_2}_{i,2}$, and

$$\begin{aligned}
& m_{13,24} \circ (\Delta \otimes \Delta)((w_1, [A, B]) \otimes (w_2, [C, D])) \\
&= m_{13,24} \left(\sum_{i,j} (w_{1i,1}, [A_{w_{1i,1}}, B_{w_{1i,1}}]) \otimes (w_{1i,2}, [A_{w_{1i,2}}, B_{w_{1i,2}}]) \right. \\
&\quad \left. \otimes (w_{2j,1}, [C_{w_{2j,1}}, D_{w_{2j,1}}]) \otimes (w_{2j,2}, [C_{w_{2j,2}}, D_{w_{2j,2}}]) \right) \\
&= \sum_{i,j} (\overline{w_{1i,1} w_{2j,1}}, [A_{w_{1i,1}} \cup C_{w_{2j,1}}^*, B_{w_{1i,1}} \cup D_{w_{2j,1}}^*]) \\
&\quad \otimes (\overline{w_{1i,2} w_{2j,2}}, [A_{w_{1i,2}} \cup C_{w_{2j,2}}^*, B_{w_{1i,2}} \cup D_{w_{2j,2}}^*]) \\
&= \sum_i (\overline{w_1 w_2}_{i,1}, [(A \cup C^*)_{\overline{w_1 w_2}_{i,1}}, (B \cup D^*)_{\overline{w_1 w_2}_{i,1}}]) \\
&\quad \otimes (\overline{w_1 w_2}_{i,2}, [(A \cup C^*)_{\overline{w_1 w_2}_{i,2}}, (B \cup D^*)_{\overline{w_1 w_2}_{i,2}}]) \\
&= \Delta \circ m((w_1, [A, B]) \otimes (w_2, [C, D])),
\end{aligned}$$

where $\Delta(w_1) = \sum_i w_{1i,1} \otimes w_{1i,2}$ and $\Delta(w_2) = \sum_j w_{2j,1} \otimes w_{2j,2}$. The last equality holds since $\overline{w_1 w_2}_{i,1}$ factor (similarly for $\overline{w_1 w_2}_{i,2}$) can be obtained by concatenating certain $w_{1i,1}$ and $w_{2j,1}$, and the indices in the intervals move along

with the words while always pointing to the same letters in Ω . We conclude that the coproduct Δ is an algebra homomorphism. Therefore, (WI, m, Δ) is a bialgebra.

Next we want to check the grading structure of (WI, m, Δ) . Let $WI_n = \text{span}_K\{(w, [A, B]) \in \mathcal{B} \mid w \text{ has } n \text{ letters in } \Omega\}$. We have $WI = \bigoplus_{n=0}^{\infty} WI_n$, so the vector space WI is graded. Specifically, we can see that $\{(w, [A, B]) \in \mathcal{B} \mid w \text{ has } 0 \text{ letters in } \Omega\} = \{(\mathcal{E}, (\emptyset, \emptyset))\}$, so $WI_0 \cong K$.

The graded piece of degree n for $WI \otimes WI$ is

$$\bigoplus_{i=0}^n WI_i \otimes WI_{n-i}$$

for any $n \in \mathbb{Z}_{\geq 0}$. For any $(w_1, [A, B]) \otimes (w_2, [C, D]) \in WI_i \otimes WI_j$ of degree $(i+j)$ where $i, j \in \mathbb{Z}_{\geq 0}$, w_1 has i letters in Ω and w_2 has j letters in Ω , so $\overline{w_1 w_2}$ in $m((w_1, [A, B]) \otimes (w_2, [C, D])) = (\overline{w_1 w_2}, [A \cup C^*, B \cup D^*])$ has $(i+j)$ letters in Ω and is of degree $(i+j)$. Hence, the multiplication map m is graded. For any $(w, [A, B]) \in WI_n$ where $n \in \mathbb{Z}_{\geq 0}$, we have

$$\Delta(w, [A, B]) = \sum_i (w_{i,1}, [A_{w_{i,1}}, B_{w_{i,1}}]) \otimes (w_{i,2}, [A_{w_{i,2}}, B_{w_{i,2}}]),$$

where $\Delta(w) = \sum_i w_{i,1} \otimes w_{i,2}$. We can see that the sum of number of letters in Ω in $w_{i,1}$ and $w_{i,2}$ is $(i+j)$ for every summand in the coproduct, so the coproduct Δ is graded. We conclude that (WI, m, Δ) is a graded connected bialgebra. By Theorem 4, (WI, m, Δ) is a graded Hopf algebra. As we have shown in Section 3.1, W_f does not have a clean expression for its antipode. (WI, m, Δ) does not have a clean expression for its antipode either, since the coproduct is still a deshuffle with respect to letters in Ω with indices adjusted accordingly.

We want to check the grading on (wI, m, δ) . Consider the coproduct

$$\delta(w, [A, B]) = \sum_{A \subset C \subset B} (w, [A, C]) \otimes (w, [C, B]),$$

for any $(w, [A, B]) \in WI$. We define the length of an interval $|[A, B]| = |B| - |A|$. If $WI = \bigoplus_{i=0}^{\infty} WI_i$, we want

$$\delta(WI_n) \subset \bigoplus_{i=0}^n WI_i \otimes WI_{n-i}$$

for any $n \in \mathbb{Z}_{\geq 0}$, and only grading on the length of the intervals would satisfy this relation, and $|A, B]| = |[A, C]| + |[C, B]|$ for every summand in the coproduct.

Let

$$WI_n = \bigoplus_{w \in \mathcal{W}_f} \text{span}_K\{(w, [A, B]) \mid [A, B] \in \text{int}(\mathcal{P}(I_w)), |[A, B]| = n\}.$$

We can see that $WI = \bigoplus_{i=0}^{\infty} WI_i$, so the vector space WI is graded. However, we have $(\mathcal{E}, [\emptyset, \emptyset]), (a_1 b_1 b_2, [\{2\}, \{2\}]), (a_1 b_1 b_2, [\{3\}, \{3\}]) \in WI_0$ and they are distinct basis elements, so $WI_0 \not\cong K$.

For any $(w_1, [A, B]) \otimes (w_2, [C, D]) \in WI_i \otimes WI_j$ of degree $(i+j)$ where $i, j \in \mathbb{Z}_{\geq 0}$, we have $|[A, B]| = i$ and $|[C, D]| = j$ and $m((w_1, [A, B]) \otimes (w_2, [C, D])) = (\overline{w_1 w_2}, [A \cup C^*, B \cup D^*])$, so

$$|[A \cup C^*, B \cup D^*]| = |B \cup D^*| - |A \cup C^*| = |B| + |D| - |A| - |C| = i + j$$

and $m((w_1, [A, B]) \otimes (w_2, [C, D]))$ is of degree $(i+j)$. We conclude that the multiplication map m is graded. Therefore, (WI, m, δ) is a graded bialgebra but not connected. We do not know whether (WI, m, δ) is a Hopf algebra.

3.3 Cointeraction

A more detailed explanation of cointeraction between bialgebras and its applications can be found in [4][5]. For the purpose of this paper, we give the definition of cointeraction between a pair of bialgebras below.

Definition 17. *A pair of bialgebras in cointeraction is a pair (A, m_A, Δ) , and (B, m_B, δ) of bialgebras with a (right) coaction $\rho : A \rightarrow A \otimes B$ of B on A such that the product, coproduct, counit, and unit of A are morphisms of right comodules. In other words,*

- $\rho(\mathcal{E}_A) = \mathcal{E}_A \otimes \mathcal{E}_B$ and for any $a, b \in A$, $\rho(ab) = \rho(a)\rho(b)$: ρ is an algebra morphism.
- $(\Delta \otimes \text{Id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta$ where $m_{1,3,24}$ is a linear map $m_{1,3,24} : A \otimes B \otimes A \otimes B \rightarrow A \otimes A \otimes B$ such that $m_{1,3,24}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = a_1 \otimes a_2 \otimes b_1 b_2$ for any $a_1, a_2 \in A$ and $b_1, b_2 \in B$.
- For any $a \in A$, $(\epsilon_A \otimes \text{Id}) \circ \rho(a) = \epsilon_A(a) \mathcal{E}_B$.

We can see that $m_{1,3,24} = (\text{Id} \otimes \text{Id} \otimes m) \otimes m_{1,3,2,4}$. Similar to our discussion on the algebra homomorphism, we have the transposition map $m_{1,3,2,4}$. The map $m_{1,3,24}$ is important in the definition of cointeraction. For any $a \in A$, we can see that $(\Delta \otimes \text{Id}) \circ \rho(a) \in A \otimes A \otimes B$ and $(\rho \otimes \rho) \circ \Delta(a) \in A \otimes B \otimes A \otimes B$. Therefore, the map $m_{1,3,24}$ is required to move the elements in A into the correct positions and multiply the elements in B together.

We can check whether the pair of bialgebras (WI, m, Δ) and (WI, m, δ) with $\rho = \delta$ are in cointeraction.

We have shown in Section 3.2 that the coproduct δ is an algebra homomorphism, so the first property is satisfied.

For any $(w, [A, B]) \in \mathcal{B}$, we have

$$(\epsilon_{\Delta} \otimes \text{Id}) \circ \delta(w, [A, B]) = \sum_{ACCBB} \epsilon_{\Delta}(w, [A, C]) \otimes (w, [C, B]),$$

so $(\epsilon_\Delta \otimes \text{Id}) \circ \delta(w, [A, B]) = 0$ if w is non-empty and $(\epsilon_\Delta \otimes \text{Id}) \circ \delta(w, [A, B]) = (\mathcal{E}, [\emptyset, \emptyset])$ if $w = \mathcal{E}$. The right hand side is 0 if w is non-empty, and $(\mathcal{E}, [\emptyset, \emptyset])$ if $w = \mathcal{E}$, so the third property is satisfied.

Now we want to check the second property. For any $(w, [A, B]) \in \mathcal{B}$, we have

$$\begin{aligned} & (\delta \otimes \delta) \circ \Delta(w, [A, B]) \\ = & \sum_{\substack{A_{w_{i,1}} \overset{i}{\subset} C \subset B_{w_{i,1}} \\ A_{w_{i,2}} \subset D \subset B_{w_{i,2}}}} (w_{i,1}, [A_{w_{i,1}}, C]) \otimes (w_{i,1}, [C, B_{w_{i,1}}]) \\ & \otimes (w_{i,2}, [A_{w_{i,2}}, D]) \otimes (w_{i,2}, [D, B_{w_{i,2}}]), \end{aligned}$$

so

$$\begin{aligned} & m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta(w, [A, B]) \\ = & \sum_{\substack{A_{w_{i,1}} \overset{i}{\subset} C \subset B_{w_{i,1}} \\ A_{w_{i,2}} \subset D \subset B_{w_{i,2}}}} (w_{i,1}, [A_{w_{i,1}}, C]) \otimes (w_{i,2}, [A_{w_{i,2}}, D]) \\ & \otimes (\overline{w_{i,1}w_{i,2}}, [C \cup D^*, B_{w_{i,1}} \cup B_{w_{i,2}}^*]), \end{aligned}$$

and we have

$$\begin{aligned} & (\Delta \otimes \text{Id}) \circ \delta(w, [A, B]) \\ = & \sum_{A \overset{i}{\subset} E \subset B} (w_{i,1}, [A_{w_{i,1}}, E_{w_{i,1}}]) \otimes (w_{i,2}, [A_{w_{i,2}}, E_{w_{i,2}}]) \otimes (w, [E, B]), \end{aligned}$$

where $\Delta(w) = \sum_i w_{i,1} \otimes w_{i,2}$. Given $E_{w_{i,1}}$ and $E_{w_{i,2}}$, we can choose $C = E_{w_{i,1}}$ and $D = E_{w_{i,2}}$, such that the first two factors in the tensor product are equal. Given C and D , we can choose E such that $E_{w_{i,1}} = C$ and $E_{w_{i,2}} = D$ such that the first two factors in the tensor product are equal. However, in the third factor of the tensor product, $\overline{w_{i,1}w_{i,2}}$ is not necessarily equal to w . For example, let $w = a_1b_1a_2$, where $\tilde{f}_w(1) = \tilde{f}_w(3) = \{2\}$ and $\tilde{f}_w(2) = \emptyset$. We have $\Delta(w) = a_1b_1a_2 \otimes \mathcal{E} + \mathcal{E} \otimes a_1b_1a_2 + a_1b_1 \otimes b_1a_2 + b_1a_2 \otimes a_1b_1$, but $a_1b_1a_2 \neq a_1b_1a_2 \neq b_1a_2a_1b_1$. In fact, if $w \in \mathcal{W}_f$ has two or more different letters in Ω , we will always have summand $w_1 \otimes w_2$ in the coproduct where w_1 and w_2 each have at least one letter in Ω and $w_1 \neq w_2$, and then $\overline{w_1w_2} = \overline{w_2w_1}$. Therefore, the pair of bialgebras (WI, m, Δ) and (WI, m, δ) is not in cointeration.

The cause of this issue is that when the deshuffle operation in the coproduct takes apart a word with respect to letters in Ω , we lose the information about how the parts fit together, so the multiplication map fails to reconstruct the original word. There are two ways in which we can resolve this issue, either by introducing an equivalence relation such that the multiplication map gives a word that is equivalent to the original word, or by introducing global information such that the multiplication map is able to give back the original word.

Let us consider the first approach. We introduce the irreducible decomposition D such that $D(\mathcal{E}, [\emptyset, \emptyset]) = (\mathcal{E}, [\emptyset, \emptyset])$ and for any $(w, [A, B]) \in \mathcal{B}$ with n

letters in Ω , where $n \in \mathbb{Z}_+$, $D((w, [A, B]))$ returns a tensor product in $\bigotimes_{i=1}^n WI$ equivalent up to permutation of factors, where each factor $(w_i, [A_{w_i}, B_{w_i}])$, which we call a singleton, contains only one letter in Ω in w_i for any $i = 1, \dots, n$. These singletons are obtained by recursively applying deshuffling coproduct to each factor of the tensor product until there is only the primitive part. By the universal property of vector spaces, the definition of D can be extended to WI . For example, let $w = a_1b_1b_2a_2b_3a_3$ where $\tilde{f}_w(1) = \{2, 3\}$, $\tilde{f}_w(4) = \{2, 5\}$, $\tilde{f}_w(6) = \{3\}$, and $\tilde{f}_{w_2}(2) = \tilde{f}_{w_2}(3) = \tilde{f}_{w_2}(5) = \emptyset$. We have

$$\begin{aligned} & D(a_1b_1b_2a_2b_3a_3, [\{3\}, \{2, 3\}]) \\ &= (a_1b_1b_2, [\{3\}, \{2, 3\}]) \otimes (b_1a_2b_3, [\emptyset, \{1\}]) \otimes (b_2a_3, [\{1\}, \{1\}]). \end{aligned}$$

Note that the irreducible decomposition D is similar to the prime factorization of integers.

We define the equivalence relation \sim such that for any $a, b \in WI$, $a \sim b$ if $D(a) = D(b)$. By the universal property of quotient spaces, all the maps discussed above are uniquely defined in the quotient space WI/\sim . With the choice of C, D, E described above, we have $(\overline{w_{i,1}w_{i,2}}, [C \cup D^*, B_{w_{i,1}} \cup B_{w_{i,2}}^*]) = (w, [E, B])$ since they have the same irreducible decomposition. With the second property also satisfied, we can claim that the pair of bialgebras $(WI/\sim, m, \Delta)$ and $(WI/\sim, m, \delta)$ is in cointeraction. We can see that this quotient structure completely subverts the structure of words. Since words corresponds to orderings of edges in a graph, this quotient structure also removes the order structure on the graph. We observe that without the order structure on the graph, our cointeraction relates back to the cointeraction by generators in [4].

The second approach is to give a global order when we define the multiplication. We want to introduce an additional restriction that for any $(w, [A, B]) \in \mathcal{B}$ each letter in Ω in w appear only once. We introduce a global order $w_g \in \mathcal{W}_f$ in which each letter in the alphabet Ω appears exactly once. We also have the global association map \tilde{f}_{w_g} that associates letters in Ω' in w_g with letters in Ω . Then all the valid words are formed by taking some letters in Ω from w_g and filling in letters from Ω' in a way that preserves the correspondence relation and the relative position. We keep the definitions of $\Delta, \delta, \epsilon_\Delta, \epsilon_\delta$ and u and change the multiplication map m . For any $(w_1, [A, B]), (w_2, [C, D]) \in RWI$ (RWI is restricted WI in the way described above), let $m((w_1, [A, B]) \otimes (w_2, [C, D])) = (w_1 \cup w_2, [(A \cup C)^\dagger, (B \cup D)^\dagger])$, where $w_1 \cup w_2$ is the word formed by taking the union of letters in Ω in w_1 and w_2 and filling in letters from Ω' in a way that preserves the correspondence relation and the relative position with respect to w_g , and $(A \cup C)^\dagger$ (or similarly $(B \cup D)^\dagger$) is the set of indices of letters in Ω' pointed to by A and C adjusted with respect to the new word $w_1 \cup w_2$. Following the similar proofs as before, we can show that (RWI, m, Δ) and (RWI, m, δ) are bialgebras.

For example, consider the graph G that we gave in 3.1 and let

$$w_g = a_1b_1b_2b_4a_3a_2b_3a_4,$$

and $\tilde{f}_{w_g}(1) = \{2, 3, 4\}$, $\tilde{f}_{w_g}(5) = \{2, 3\}$, $\tilde{f}_{w_g}(6) = \{4, 7\}$, $\tilde{f}_{w_g}(8) = \{3, 7\}$, and

$\tilde{f}_{w_g}(2) = \tilde{f}_{w_g}(3) = \tilde{f}_{w_g}(4) = \tilde{f}_{w_g}(7)$. We have $m((a_1b_1b_2b_4a_2b_3, [\{2\}, \{2, 3\}]) \otimes (b_1b_2b_4a_3a_2b_3, [\{6\}, \{1, 6\}])) = (a_1b_1b_2b_4a_3a_2b_3, [\{2, 7\}, \{2, 3, 7\}])$.

With this new definition of the multiplication map m , we have

$$m((w_{i,1}, [C, B_{w_{i,1}}]) \otimes (w_{i,2}, [D, B_{w_{i,2}}])) = (w_{i,1} \cup w_{i,2}, [(C \cup D)^\dagger, (B_{w_{i,1}} \cup B_{w_{i,2}})^\dagger]).$$

By our choice of C, D, E before, we have $m((w_{i,1}, [C, B_{w_{i,1}}]) \otimes (w_{i,2}, [D, B_{w_{i,2}}])) = (w, [E, B])$, so the second property is satisfied. Since the definitions of $\Delta, \delta, \epsilon_\Delta, \epsilon_\delta$ and u are kept the same, the first and the third properties still hold. Therefore, we conclude that the pair of bialgebras (RWI, m, Δ) and (RWI, m, δ) are in cointeraction.

Note that with the modification, the bialgebras in both approaches become commutative. The commutativity is required for the third factors in the tensor products to be equal. Compare to the first approach, the second approach preserves the order structure on the graph. However, if the alphabets are big, we would have to maintain an extremely long word w_g .

The multiplication, coproduct, unit, and counit maps of the bialgebras mentioned in this paper can be summarized into the following two tables.

	Multiplication	Coproduct
Concatenation-Deshuffle	Concatenation	Deshuffle
Shuffle-Deconcatenation	Shuffle	Deconcatenation
C-D on $\Omega \cup \Omega'$	Concatenation	Deshuffle the Ω letters Carry along Ω' letters
(WI, m, δ)	Concatenate the words Union the sets in the intervals	Carry along the word Interval incidence coproduct
(WI, m, Δ)	Concatenate the words Union the sets in the intervals	Deshuffle the word Adjust the intervals
$(WI/\sim, m, \delta)$	Concatenate the words Union the sets in the intervals	Carry along the word Interval incidence coproduct
$(WI/\sim, m, \Delta)$	Concatenate the words Union the sets in the intervals	Deshuffle the word Adjust the intervals
(RWI, m, δ)	Fit subwords into the global order Adjust the intervals	Carry along the word Interval incidence coproduct
(RWI, m, Δ)	Fit subwords into the global order Adjust the intervals	Deshuffle the word Adjust the intervals
	Unit	Counit
Concatenation-Deshuffle	$u(1) = \mathcal{E}$	$\epsilon(\mathcal{E}) = 1, \epsilon(w) = 0$ for non-empty w
Shuffle-Deconcatenation	$u(1) = \mathcal{E}$	$\epsilon(\mathcal{E}) = 1, \epsilon(w) = 0$ for non-empty w
C-D on $\Omega \cup \Omega'$	$u(1) = \mathcal{E}$	$\epsilon(\mathcal{E}) = 1, \epsilon(w) = 0$ for non-empty w
(WI, m, δ)	$u(1) = (\mathcal{E}, \emptyset, \emptyset)$	$\epsilon_\delta(w, [A, B]) = 1$ if $A = B, 0$ otherwise
(WI, m, Δ)	$u(1) = (\mathcal{E}, \emptyset, \emptyset)$	$\epsilon_\Delta(w, [A, B]) = 1$ if $w = \mathcal{E}, 0$ otherwise
$(WI/\sim, m, \delta)$	$u(1) = (\mathcal{E}, \emptyset, \emptyset)$	$\epsilon_\delta(w, [A, B]) = 1$ if $A = B, 0$ otherwise
$(WI/\sim, m, \Delta)$	$u(1) = (\mathcal{E}, \emptyset, \emptyset)$	$\epsilon_\Delta(w, [A, B]) = 1$ if $w = \mathcal{E}, 0$ otherwise
(RWI, m, δ)	$u(1) = (\mathcal{E}, \emptyset, \emptyset)$	$\epsilon_\delta(w, [A, B]) = 1$ if $A = B, 0$ otherwise
(RWI, m, Δ)	$u(1) = (\mathcal{E}, \emptyset, \emptyset)$	$\epsilon_\Delta(w, [A, B]) = 1$ if $w = \mathcal{E}, 0$ otherwise

4 Conclusion

By exploring the pair (WI, m, Δ) and (WI, m, δ) from the incidence structure of concatenation-deshuffle bialgebra on two alphabets, we find that the two bialgebras are unfortunately not in cointeraction. Specifically, commutativity is necessary for the cointeraction of such word bialgebras, and we also lose information about the relative position when deshuffling words with respect to letters in Ω . The two solutions, namely, introducing a quotient structure such

that the information about relative position no longer matters, or including global information to allow multiplication to construct the original word from summands of coproduct, each yield a way of making two word bialgebras in cointeraction.

Although we introduced the concept of Hopf algebra in Section 2, we only need bialgebras in the definition of cointeractions. Yet many of the bialgebras from combinatorial problems are indeed Hopf algebras and the antipode map is related to Möbius inversion in incidence algebra [3] and renormalization in Feynmann graphs [2]. As we have shown in Section 2 and Section 3, finding the antipode can often be difficult and it may not have a clean expression. Since (WI, m, δ) is not a graded connected bialgebra, we do not even know whether it is a Hopf algebra, and constructing its antipode might be an interesting research direction.

The original motivation of studying (WI, m, Δ) and (WI, m, δ) was to add an order structure to the graph. One immediate question that we can ask is what will be the impact of the cointeractions that we constructed in two different ways. As Kreimer and Yeats explored in [4], the order on the graph comes from the sector decomposition of Feynmann integrals. The coproduct Δ on WI corresponds to taking out subgraphs as in renormalization Hopf algebras, because we take out the union of fundamental cycles that forms the subgraph and only bridgeless subgraphs are relevant for renormalization Hopf algebras. As we have discussed in Section 3.2, the incidence structure induces cuts inside the graph. These cuts show up in understanding the monodromy of singularities of Feynman integrals, so the cointeraction shows how renormalization and monodromy work well together. One way to do Feynman integrals is by sector decomposition, where we break up the region of integration based on ordering the edges, we want to put an order on the edges. By developing the cointeraction Kreimer and Yeats constructed in [4], we can see that sector decomposition is also compatible with how the renormalization and monodromy work together, albeit with additional restrictions.

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