Combinatorial approaches in quantum field theory

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## Combinatorics providing insights in QFT

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Many of these have appeared in journals now. And there are many more.
Simple nestings and chainings

Today there’s only time to talk about one of these, so I will talk about Dyson-Schwinger equations.

An example in Yukawa theory (Broadhurst-Kreimer arXiv:hep-th/0012146)

\[
G(x, L) = 1 - \frac{x}{q^2} \int d^4 k \frac{k \cdot q}{k^2 G(x, \log k^2/\mu^2)(k + q)^2} - \cdots \bigg|_{q^2=\mu^2}
\]

where \( L = \log(q^2/\mu^2) \).

How to capture the combinatorics of the recursion?

\[
X(x) = \mathbb{I} - xB_+ \left( \frac{1}{X(x)} \right)
\]
Combinatorial Dyson-Schwinger equations

We can capture other recursions in a similar language – this is equivalent to the diagrammatic viewpoint on Dyson-Schwinger equations.

Eg QED:
Putting the analysis back in

In the Yukawa example we had

\[ G(x, L) = 1 - \frac{x}{q^2} \int d^4k \frac{k \cdot q}{k^2 G(x, \log k^2/\mu^2)(k + q)^2} - \cdots |_{q^2=\mu^2} \]

- plug in \( G(x, L) = 1 - \sum \gamma_k(x)L^k \)
- use \( \partial^k_x x^{-\rho} |_{\rho=0} = (-1)^k \log^k(x) \)
- switch the order of \( \int \) and \( \partial \)

To obtain

\[
\boxed{G(x, L) = 1 - xG(x, \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)|_{\rho=0}}
\]

Where \( F(\rho) \) is the integral for the primitive regularized by a parameter \( \rho \) which marks the insertion place.
Today’s analytic Dyson-Schwinger equations

Beginning with a combinatorial Dyson-Schwinger equation

\[ X = I \pm \sum_{k \geq 1} x^k B_+^{\gamma_k} (X Q^k) \]

where \( Q = X^{-s} \), define the analytic Dyson-Schwinger equation of to be

\[ G(x, L) = 1 \pm \sum_{k \geq 1} x^k G(x, \partial_{-\rho})^{1-s^k} (e^{-L\rho} - 1) F_k(\rho)|_{\rho=0} \]

where \( F_k(\rho) \) is the Feynman integral for \( \gamma_k \) regularized by a parameter \( \rho \) which marks the insertion place.

More insertion places and systems get more complicated.
Rearranging Dyson-Schwinger equations

The Yukawa example is particularly nice and can in fact be solved.

This example works so well because the Dyson-Schwinger equation had

- One primitive graph
- which had a particularly nice integral (scaled just a geometric series)
- inserted into one place

The program of arXiv:0810.2249, Memoir. Am. Math. Soc. 211, no. 995, with an important improvement in arXiv:1302.0080, was to generalize this nice situation into a general reduction process for Dyson-Schwinger equations.

Some steps make combinatorial sense, others do not.
Finding the $\gamma_k$ recurrence

Write

$$G(x, L) = 1 \pm \sum_{k \geq 1} \gamma_k(x) L^k$$

We can find a recurrence for $\gamma_k$ in terms of lower $\gamma_j$ – it is the renormalization group equation translated into this language:

$$\left( \frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} \pm \gamma_1(x) \right) G(x, L) = 0$$

Extracting the coefficient of $L^{k-1}$ gives a recurrence for $\gamma_k$

$$\gamma_k = \frac{1}{k} \gamma_1(x) (-\text{sign}(s) + |s|x\partial_x) \gamma_{k-1}(x)$$

for $k \geq 2$
Trading $\rho$ for $x$

Notice that $\gamma_k(x)$ begins with an $x^k$ term. So the lowest possible power of $x$ in

$$x^k G(x, \partial_{-\rho})^{1-sk} \rho^{\ell} \big|_{\rho=0}$$

is

Consequently there is a unique sequence $r_k$ such that

$$\sum_k x^k G(x, \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) F_k(\rho) \bigg|_{\rho=0}$$

$$= \sum_k x^k G(x, \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) \frac{r_k}{\rho(1-\rho)} \bigg|_{\rho=0}$$
The differential equation

Taking the coefficient of $L$ and $L^2$ in

$$G(x, L) = 1 \pm \sum_k x^k G(x, \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) \frac{r_k}{\rho(1 - \rho)} \bigg|_{\rho=0}$$

and then using the $\gamma_k$ recurrence we get

$$\gamma_1(x) = -P(x) + \gamma_1(x)(\text{sign}(s) - |s x \partial_x) \gamma_1(x)$$

where

$$P(x) = \sum_{k \geq 1} r_k x^k$$
The differential equation in QED

Joint work with Guillaume van Baalen, Dirk Kreimer, and David Uminsky, arXiv:0805.0826.

In QED in the Baker, Johnson, Willey gauge, we only need to worry about the photon, so we are in the single equation case.

\[ s = 1 \text{ because} \]
There are two behaviours. The *separatrix* is the separating solution.

\$s=1, \ P=x\$
Results

If \( P(x) \) is \( C^2 \) and \( P(x) > 0 \) for \( x \in (0, x_0) \) then either

- \( \gamma_1 \) crosses the \( x \) axis with a vertical tangent and returns to \(-1\), or
- \( P \) and \( \gamma_1 \) have a common zero, or
- \( \gamma_1 \) is positive and exists for all \( x \)

In the last case if also \( P(x) > 0 \) for all \( x > 0 \) and \( P(x) \) is increasing then either

- \( \gamma_1 \) is the separatrix and diverges in finite \( L \) (a Landau pole) iff
  \[
  \int_{x_0}^{\infty} \frac{2dz}{z(\sqrt{1 + 4P(z)} - 1)} < \infty
  \]
- \( \gamma_1 \) is larger than the separatrix and diverges in finite \( L \) regardless of \( P \).
Other results

We also thought about other values of $s$ including in arXiv:0906.1754 negative values of $s$ which have quite a different flavour (spirals!) and form a model of massless QCD.

Looking at $s = 2$ we can give an explicit combinatorial solution as a sum over rooted connected chord diagrams

Marc Bellon and his collaborators have looked at the Wess-Zumino model, eg arXiv:1205.0022, and specific approximations to $P$.