A combinatorial perspective on Dyson-Schwinger equations

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An equation

$$\gamma_1(x) = P(x) - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x)$$

Important special cases

$$\gamma_1(x) = x - \gamma_1(x)(1 - 2x\partial_x)\gamma_1(x)$$
$$2\gamma_1(x) = \left(\frac{x}{3} + \frac{x^2}{4}\right) - \gamma_1(x)(1 - x\partial_x)\gamma_1(x)$$

$$\gamma_1^+(x) = P^+(x) - \gamma_1^+(x)^2 - (\gamma_1^+(x) - 2\gamma_1^-(x))x\partial_x\gamma_1^+(x)$$
$$\gamma_1^-(x) = P^-(x) - \gamma_1^-(x)^2 - (\gamma_1^+(x) - 2\gamma_1^-(x))x\partial_x\gamma_1^-(x)$$

The Hopf algebra

As a vector space the $\mathbb{Q}$ span of Feynman graphs with $\emptyset$ the empty graph.

As an algebra with multiplication $m$ given by disjoint union.

As a coalgebra with coproduct

$$\Delta(\Gamma) = \sum_{\gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma$$

$$\gamma$$ product of divergent 1PI subgraphs

As a Hopf algebra with antipode defined recursively by $S(\emptyset) = \emptyset$,

$$S(\Gamma) = -\Gamma - \sum_{\emptyset \neq \gamma \subseteq \Gamma} S(\gamma)\Gamma/\gamma$$

$$\gamma$$ product of divergent 1PI subgraphs

on connected graphs, and extended as an anti-homomorphism.

Feynman diagrams

Said to be divergent if they have one of a given set of external leg structures (for a renormalizable theory).
Said to be 1-particle irreducible (1PI) if they are 2-edge connected.
Example

\[ \Delta \left( \begin{array}{c} \circ \otimes \circ \otimes \circ \otimes \circ \end{array} \right) = \begin{array}{c} \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \end{array} + \begin{array}{c} \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \end{array} + 2 \begin{array}{c} \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \end{array} . \]

A Dyson-Schwinger equation

Consider

\[ X(x) = \mathbb{1} - \sum_{k \geq 1} x^k B_+^k (X(x)Q(x)^k) \]

where \( Q(x) = X(x)^{-s} \) with \( s > 0 \) an integer. Associate with each \( B_+ \) a Mellin transform

\[ F_k \left( \rho_1, \ldots, \rho_n \right) . \]

Write the combination \( (X \mapsto G, B_+^k \mapsto F^k) \), the \( \rho_i \) mark the insertion places) as

\[ G(x, L) = \sum \gamma_k(x) L^k \quad \text{with} \quad \gamma_k(x) = \sum_{j \geq k} \gamma_{k,j} x^j . \]

Systems of equation are similar but messier.

\[ B_+ \]

Write \( B_+^\gamma \) for insertion into the primitive graph \( \gamma \). For example

\[ B_+ \left( \begin{array}{c} \circ \otimes \circ \otimes \circ \otimes \circ \end{array} \right) = 2 \begin{array}{c} \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \otimes \circ \end{array} . \]

The \( B_+ \) are Hochschild 1-cocycles

\[ \Delta B_+ = (\text{id} \otimes B_+) \Delta + B_+ \otimes \mathbb{1} . \]

An example

Broadhurst and Kreimer; a bit of massless Yukawa theory.

\[ X(x) = \mathbb{1} - x B_+ \left( \frac{1}{X(x)} \right) , \]

\[ F(\rho) = \frac{1}{q^2} \int d^4 k \frac{k \cdot q}{(k^2)^{1+\epsilon}(k + q)^2} - \cdots \bigg|_{q^2 = \mu^2} . \]

Combine to get

\[ G(x, L) = 1 - \frac{x}{q^2} \int d^4 k \frac{k \cdot q}{k^2 G(x, \log k^2)(k + q)^2} - \cdots \bigg|_{q^2 = \mu^2} , \]

where \( L = \log(q^2 / \mu^2) \).
Four initial steps

1. From the renormalization group equation or the scattering type formula of Connes and Kreimer derive

\[ \gamma_k(x) = \frac{1}{k} \gamma_1(x)(1 - sx\partial_x)\gamma_{k-1}(x), \]

2. Reduce to primitives with a single insertion place, that is Mellin transforms in a single variable \( \rho \). For example with

\[ X = 1 - xB^{\frac{1}{2}}_1 - \frac{1}{X^2} \]

use

\[ q_1 = \frac{1}{2}, \quad q_2 = 0, \quad q_3 = \frac{1}{8} - \frac{1}{8} \]

3. Rewrite the (analytic) Dyson-Schwinger equation using the usual tricks

- plug in \( \sum \gamma_k L^k \)
- use \( \partial_\rho x^{-\rho}|_{\rho=0} = (-1)^k \log^k(x) \)
- switch the order of \( \int \) and \( \partial \)

\[ \gamma \cdot L = \sum x^k(1 + \gamma \cdot \partial_\rho)^{sk+1}(1 - e^{-L\rho})F^k(\rho) \bigg|_{\rho=0} \]

where \( \gamma \cdot U = \sum \gamma_k U^k \).

4. Reduce to geometric series Mellin transforms by noticing that for \( \ell \geq 0 \)

\[ x^k(1 + \gamma \cdot \partial_\rho)^{-rk+1}\rho^\ell|_{\rho=0} \]

viewed as a series in \( x \) has lowest term \( x^{k+\ell} \).

Finding \( \gamma_1 \)

From step 3

\[ \gamma \cdot L = \sum x^k(1 + \gamma \cdot \partial_\rho)^{sk+1}(1 - e^{-L\rho})F^k(\rho) \bigg|_{\rho=0}. \]

Take an \( L \) derivative and set \( L = 0 \) to get

\[ \gamma_1 = \sum x^k(1 + \gamma \cdot \partial_\rho)^{sk+1}F^k(\rho) \bigg|_{\rho=0}. \]

Take two \( L \) derivatives and use step 4 to get

\[ 2\gamma_2 = - \sum x^k(1 + \gamma \cdot \partial_\rho)^{sk+1}r_k \rho \bigg|_{\rho=0} = -\gamma_1 + \sum x^k r_k. \]

Write \( P(x) = \sum x^k r_k \) and use step 1

\[ \gamma_1 = P(x) - \gamma_1(1 - sx\partial_x)\gamma_1. \]

As a recursive equation

View

\[ \gamma_1(x) = P(x) - \gamma_1(x)(1 - sx\partial_x)\gamma_1(x) \]

as a recursive equation. At the level of coefficients

\[ \gamma_{1,n} = p(n) + \sum_{j=1}^{n-1}(-rj-1)\gamma_{1,j}\gamma_{1,n-j}. \]

Assume \( \gamma_{1,1} \neq 0 \) and \( f(x) = \sum \frac{p(n)x^n}{n!} \) has radius of convergence \( \rho > 0 \). Let \( a(n) = \frac{\gamma_{1,n}}{n!} \). The recursion becomes

\[ a_n = \frac{p(n)}{n!} + \sum_{i=1}^{n-1}(-ri-1)a_ia_{n-i} \left( \frac{n}{i} \right)^{-1} \]

\[ = \frac{p(n)}{n!} + \left( -\frac{rn}{2} - 1 \right) \sum_{i=1}^{n-1} a_ia_{n-i} \left( \frac{n}{i} \right)^{-1}. \]
How bad is the growth of $\gamma_1$?

Idea:
\[
an(n) \text{ is approximately } \frac{p(n)}{n!} - ra_1a_{n-1}
\]
giving a radius of min \( \left\{ \rho, \frac{1}{-ra_1} \right\} \) for \( \sum a_n x^n \). Implement the idea by bounding on each side.

Easy direction:
\[
a_n \geq \frac{p(n)}{n!} - r \frac{n-2}{n} a_1 a_{n-1}
\]

Messy direction: for any \( \epsilon > 0 \) there is an \( N > 0 \) such that for \( n > N \)
\[
a_n \leq \frac{p(n)}{n!} - ra_1a_{n-1} + \epsilon \sum_{j=1}^{n-1} a_j a_{n-j}
\]

Why?

- Understanding the growth of \( \gamma_1 \) is understanding the growth of the whole theory.
- Expect a Lipatov bound \( \gamma_{1,n} \leq c^n n! \).
- Does the first singularity of \( \sum \frac{\gamma_{1,n}}{n!} x^n \) come from renormalon chains or from instantons?
- We’ve shown that a Lipatov bound for the primitives leads to a Lipatov bound on the whole theory.
- The radius is either the radius from the primitives or \( \frac{1}{r \gamma_{1,1}} \), the first coefficient of the beta function.
- The moral is that the primitives control matters.

As a differential equation

View
\[
\gamma_1(x) = P(x) - \gamma_1(x)(1-sx\partial_x)\gamma_1(x)
\]
as a differential equation.

The \( P(x) = x \) family is the last bastion of exact solutions,

\[s = 1: \quad \gamma_1(x) = x + x W \left( C \exp \left( -\frac{1+x}{x} \right) \right), \]

\[s = 2: \quad \exp \left( \frac{(1+\gamma_1(x))^2}{2x} \right) \sqrt{-x} + \text{erf} \left( \frac{1+\gamma_1(x)}{\sqrt{2x}} \right) \frac{\sqrt{x}}{\sqrt{2}} = C \]

\[s = 3/2: \quad A(X) - x^{1/3} 2^{1/3} A'(X) = C \left( B(X) - x^{1/3} 2^{1/3} B'(X) \right) \text{ where } X = \frac{1+\gamma_1(x)}{2^{7/3} x^{2/3}} \]

\[s = 3: \quad (\gamma_1(x)+1) A(X) - 2^{2/3} A'(X) = C \left( (\gamma_1(x)+1) B(X) - 2^{2/3} B'(X) \right) \text{ where } X = \frac{(1+\gamma_1(x))^2 + 2x}{2^{7/3} x^{2/3}} \]

where \( A \) is the Airy Ai function, \( B \) the Airy Bi function and \( W \) the Lambert W function.
QED as a single equation

By the Baker, Johnson, Willey analysis we can reduce to a single equation for the photon propagator.

\[ 2 \gamma_1(x) = P(x) - \gamma_1(x)(1 - x\partial_x)\gamma_1(x) \]

\( s = 1 \) gives a term \( B_+ (\mathbb{1}) \) independent of \( X \) to take into account the fact that the photon propagator cannot be inserted into the one loop graph.

To 2 loops

\[ P(x) = \frac{x}{3} + \frac{x^2}{4} \]

To 4 loops we need to correct the primitives for our setup

\[ P(x) = \frac{x}{3} + \frac{x^2}{4} + (-0.0312 + 0.06037)x^3 + (-0.6755 + 0.05074)x^4 \]
QED to 2 loops

At 4 loops $P(0.992\ldots) = 0$ changing everything.

Zoomed in

We are here