Dyson-Schwinger equations and Renormalization Hopf algebras

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Unfolding some recursive equations

Let’s get our intuition going

\[ X = \mathbb{1} + x B_+(X^2) \]

What does this count?

\[ X = \mathbb{1} + x B_+(X^3) \]

What does this count?

\[ X = \mathbb{1} - x B_+ \left( \frac{1}{X} \right) \]

What does this count?

Answers

\[ X = \mathbb{1} + x B_+(X^2) \]

counts computer science binary trees (separate slots for left and right children).

\[ X = \mathbb{1} + x B_+(X^3) \]

counts ternary trees with separate slots for left, middle, and right children.

\[ X = \mathbb{1} - x B_+ \left( \frac{1}{X} \right) \]

counts plane rooted trees.

Dyson-Schwinger equations combinatorially

As the simple tree examples, or systems

\[ X^r(x) = \mathbb{1} - \sum_{k \geq 1} x^k p_r(k) B_+^{k,r} (X^r(x) Q(x)^k) \]

where \( Q(x) = \prod X^r(x)^{s_r} \) and \( r \) runs over the different external leg structures.

Example: QED

\[ \cdots \]
Dyson-Schwinger equations analytically

Example from Broadhurst and Kreimer [3].
\[ X(x) = 1 - xB_+ \left( \frac{1}{X(x)} \right). \]
along with
\[ F(\rho) = \frac{1}{q^2} \int d^4k \frac{k \cdot q}{(k^2)^{1+\rho}} \frac{k \cdot q}{k^2G(x, \log k^2)(k + q)^2} - \cdots |_{q^2=\mu^2} \]
gives \((X \mapsto G, B_+ \mapsto F)\)
\[ G(x, L) = 1 - x \frac{q^2}{q^2} \int d^4k \frac{k \cdot q}{k^2G(x, \log k^2)(k + q)^2} \]
\[ - \cdots |_{q^2=\mu^2} \]
where \(L = \log(q^2/\mu^2)\). The (analytic) Dyson-Schwinger equation for a bit of massless Yukawa theory.

Dyson-Schwinger equations physically

Equations of motion, analogous to the classical differential equations of motion.

By expanding in the coupling constant Dyson-Schwinger equations give perturbation theory.

But Dyson-Schwinger equations also contain non-perturbative information if we can extract it. Broadhurst and Kreimer [3] solved
\[ G(x, L) = 1 - x \frac{q^2}{q^2} \int d^4k \frac{k \cdot q}{k^2G(x, \log k^2)(k + q)^2} \]
\[ - \cdots |_{q^2=\mu^2} \]
where \(L = \log(q^2/\mu^2)\) parametrically with
\[ G(x, L) = \frac{\sqrt{\pi}}{\exp(p^2) \text{erfc}(p)} q^2 = \mu^2 \left( \frac{\text{erfc}(p)}{\text{erfc}(p_0)} \right)^{1/2} \]
Other physical perspectives: http://web.mit.edu/redingtn/www/netadv/Xdysonschw.html

Dyson-Schwinger equations and \(B_+\)

The key is \(B_+\).

All the Hopf algebras we’re interested in are generated by one or more \(B_+\) and so are the solutions of Dyson-Schwinger equations or quotients thereof.

\(B_+\) is a 1-cocycle
\[ \Delta B_+ = (\text{id} \otimes B_+)\Delta + B_+ \otimes \mathbb{I} \]
A subpiece comes from the branches, or is the whole thing. Unique decomposition.

\((\mathcal{H}_{rt}, B_+)\) is universal for Hopf algebras with a 1-cocycle. Connes, Kreimer: [4].

\(B_+\) and the universal law

The 1-cocycle property is the cohomological way to say unique decomposition.

Rooted trees are nice due to the unique decomposition of a tree into its root and the forest of its subtrees: \(B_+\). For unlabelled trees, \(T(x) = \sum t(n)x^n\),
\[ T(x) = x \exp \left( \sum_{m \geq 1} \frac{T(x^m)/m}{m} \right). \]
Which by Pólya’s classical analysis gives the asymptotics
\[ t(n) \sim C \rho^{-n} n^{-3/2} \]
Asymptotics of the form \(C \rho^{-n} n^{-3/2}\) are ubiquitous for classes of rooted trees with recursive definitions, hence the term universal law.
Operators giving the universal law

How ubiquitous? Let $O$ be the set of operators on power series built out of $E(x, \cdot)$ such that

(a) $E(x, y)$ has nonnegative coefficients and zero constant term,
(b) $E(a, b) < \infty \Rightarrow \exists \epsilon > 0, E(a + \epsilon, b + \epsilon) < \infty$,
(c) $\exists R > 0, [x^iy^j]E(x, y) \leq R^{i+j}$.

1. $MSet_M$ and $Seq_M$ for all $M \subseteq \mathbb{Z}^{>0}$.
2. $DCycle_M$ and $Cycle_M$ for $\sum_{m \in M} 1/m = \infty$ or $M$ finite.

Using scalar multiplication from $\mathbb{R}^{\geq0}$, addition, multiplication, and composition, and where if $MSet_M$, $DCycle_M$, or $Cycle_M$ appear then scalars and coefficients of $E$ must be integers.

Theorem 1. [Bell, Burris, –[1]] Let $\Theta \in O$ such that

- $\Theta$ is nonlinear
- $[x^n]\Theta(A(x))$ depends only on $[x^i]A(x)$ for $i < n$.

Let $A(x)$ be a power series

- with nonnegative coefficients
- with zero constant term
- which diverges at its radius of convergence
- if $MSet_M$, $DCycle_M$, or $Cycle_M$ appear in $\Theta$ then $A(x)$ has integer coefficients.

Then there is a unique $T(x)$ satisfying

$$T(x) = A(x) + \Theta(T)(x).$$

The coefficients of $T$ satisfy the universal law on their support.

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$B_+$ and the first recursion

For an analytic Dyson-Schwinger equation write

$$G(x, L) = \sum \gamma_k(x)L^k \quad \gamma_k = \sum_{j \geq k} \gamma_{k,j}x^j$$

The Hochschild closedness of $B_+$ is what permits us to rewrite the linearized coproduct which along with $S \ast Y$ gives the recursion ([5])

$$\gamma_k(x) = \frac{1}{k} \gamma_1(1 + rx\partial_x)\gamma_{k-1}(x)$$

$B_+$ and the second recursion

Again

$$G(x, L) = \sum \gamma_k(x)L^k \quad \gamma_k = \sum_{j \geq k} \gamma_{k,j}x^j$$

The properties of $B_+$ don’t care about connectedness which permits us to modify the primitives of the theory to

- reduce to one insertion place; univariate Mellin transforms.
- take away higher order behaviour of Mellin transforms; geometric series Mellin transforms.

which along with the other recursion gives ([6])

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1} (-rj - 1)\gamma_{1,j}\gamma_{1,n-j}$$
**B+ and the growth of $\gamma_1$**

$$\gamma_{1,n} = p(n) + \sum_{j=1}^{n-1} (-rj - 1)\gamma_{1,j}\gamma_{1,n-j}$$

is what we were able to analyze to show that the primitives determine the growth of the whole theory.

In particular Lipatov bounds $\gamma_{1,n} \leq c^n n!$ carry over.

**B+ and sub Hopf algebras**

Today’s punchline, solutions to Dyson-Schwinger equations are sub Hopf algebras. Bergbauer, Kreimer [2].

In the example

$$X = 1 + xB_+(X^2)$$

$$c_0 = \varepsilon \quad c_1 = * \quad c_2 = 2 \varepsilon \quad c_3 = \Lambda + 4\varepsilon$$

check

$$\Delta c_4 = 4\left( \gamma_{\varepsilon} \gamma_{\varepsilon} + \varepsilon \gamma_{\varepsilon} \gamma_{\varepsilon} \varepsilon \right) + 2\left( \gamma_{\Lambda} \gamma_{\varepsilon} \gamma_{\varepsilon} \varepsilon \right) + \gamma_{\varepsilon} \gamma_{\varepsilon} \gamma_{\varepsilon} \gamma_{\varepsilon} \varepsilon$$

$$= c_4 \otimes c_0 + c_0 \otimes c_4 + (2c_3 + 2c_2) \otimes c_1 + (3c_2 + 3c_3) \otimes c_0$$

**The sub Hopf algebra result**

Let $B^{d+}_+$ be Hochschild 1-cocycles. Consider

$$X = 1 + \sum x^n w_n B^{d+}_+ (X^{n+1})$$

write $X = \sum x^n c_n$. Then the Dyson-Schwinger equation has a unique solution

$$c_n = \sum w_m B^{d+}_+ \sum_{k_1 + \cdots + k_m = n-m} c_{k_1} \cdots c_{k_{m+1}}$$

and the $c_n$ generate a sub Hopf algebra

$$\Delta c_n = \sum_{k=0}^{n} P_k^n \otimes c_k$$

where the $P_k^n$ are homogeneous polynomials of degree $n-k$ in the $c_i$, specifically

$$P_k^n = \sum_{\ell_1 + \cdots + \ell_{k+1}} c_{\ell_1} \cdots c_{\ell_{k+1}}$$

**The role of $B_+$ for the sub Hopf algebras**


The inductive proof has the advantage of showing explicitly the use of the Hochschild 1-cocycle property of $B_+$ and that no deep facts are needed.
References


