Quantum field theory, algebraic geometry, and graph theory

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Numbers in Feynman integrals.

Interesting numbers show up in perturbative quantum field theory calculations.

These *interesting numbers* include

- Multiple zeta values
- Evaluations of multiple polylogarithms
- Elliptic polylogarithms
- And more
What are these calculations?

- In perturbative quantum field theory you calculate physical things using series indexed by Feynman diagrams – certain graphs describing particle interactions.
- Each Feynman diagram contributes an integral where the integrand is built out of pieces corresponding to edges and vertices in the graph.
- Or you can try to be more clever and index the sum in other ways.
Where do the interesting numbers appear?

These numbers appear almost no matter what you do.

- These numbers appear in Feynman integrals. Which graphs produce which integrals?
- They appear in essentially any interesting choice of quantum field theory.
- They appear in the sum (not just in the individual graphs).
- They appear in other approaches to perturbative quantum field theory.
- There are patterns to how they appear, but it is still mysterious.
Why do the numbers appear?

Why indeed?

There must be some good mathematical reason.

That would be interesting mathematics and it would explain something about Feynman integrals and hence let us do more physics too.
Keep it simple

One example of a quantum field theory is $\phi^4$

one edge type —

one vertex type + all vertex deg 4 (but external edges are allowed)

eg $\ \ \ \ \ \ \ $ etc.

These will be our graphs today.
The Kirchhoff polynomial

Let $K$ be a connected 4-regular graph.
Let $G = K - v$. These are connected $\phi^4$ graphs with 4 external edges.

Define

$$\psi_G = \sum \prod_{T \in \mathcal{S}} a_e$$

associate a variable to each edge of the graph, $a_e$ for edge $e$

running over spanning trees

Eg:

$$\psi = bd + bc + ad + ac + cd$$
The Feynman period

Rather than consider the full Feynman integral and what numbers and functions of the parameters show up, we’ll just think about:

\[ P_G = \int \frac{\Omega}{\Psi^2} \text{ projective form} \]

\[ \Omega = \sum_{\ell \in \mathbb{C}} \tilde{\ell} \text{d} a_1, \ldots, \text{d} a_n \ldots \]

Eg:

\[ \psi = a + b \quad P = \int_{a=0}^{\infty} \frac{\text{d} a}{(a+1)^2} \]

more excitly \[ \prod_{k} k_4 \quad P_{k_4} = 6 \mathcal{F}(3) \]
Sketch of Brown’s approach to integration

Francis Brown gave an algorithm for how to integrate some of these. There will be multiple polylogarithms in the numerator and polynomials in the denominator.

\[ \sum \frac{1}{\psi_G} \rightarrow \int \frac{1}{\psi_{G \cdot e_1} \psi_{G / e_1}} \]

\[ \rightarrow \int \frac{\text{logs}}{\text{poly}} \]

\[ \rightarrow \int \frac{\text{more logs}}{\text{poly}} \]

\[ \rightarrow \int \frac{\text{dilogs}}{\text{poly}} \rightarrow \int \frac{\text{trilogs}}{\text{poly}} \]
You can’t always continue because the polynomial here doesn’t necessarily factor.

In general

\[ \int \frac{\text{polylog of weight } n}{\text{poly}} \]

if poly factors into distinct linear factors

get \[ \int \frac{\text{polylog of weight } n!}{\text{disc}} \]

if square in denominator lower weight

if doesn’t factor die.
Period – geometry – arithmetic

\[ \frac{1}{\text{Sup}} \text{ should be controlled by the geometry of } \psi = 0 \]

we have a different perspective on this geometry by counting points over finite fields \( \psi = 0 \)
The $c_2$ invariant

For $f \in \mathbb{Z}[x_1, \ldots, x_n]$ define $[f]_q$ to be the number of $\mathbb{F}_q$-rational points on the variety $f = 0$.

Define

$$c_2^{(p)}(G) = \frac{[4_0]_p}{p^2} \mod p$$

Note if $[4_0]_p$ were polynomial in $p$
then $c_2^{(p)}(G)$ would be the quadratic coeff and so independent of $p$.
Arithmetic structure

- If $c_2^{(p)}(G)$ is independent of $p$ then $P_G$ should be MZV.
- If $c_2^{(p)}(G) = 0$ then $P_G$ should have less than maximal transcendental weight.
- If $c_2^{(p)}(G)$ is constant in some field extension then $P_G$ should be a multiple polylogarithm evaluated at the roots.
- Some $c_2^{(p)}(G)$ are proven to be coefficient sequences of modular forms.
- In this case $P_G$ should be more exotic.
Known graph-related properties

- If $K$ has a 3-separation then $c_2^{(p)}(G) = 0$.
- If $K$ has an internal 4-edge-cut then $c_2^{(p)}(G) = 0$.
- If $G$ has vertex width 3 then $c_2^{(p)}$ is a constant.
- $c_2$ is double-triangle invariant.

$G$ has vertex width 3 if more than one vertex is on each side of the cut.
Known and conjectured symmetries

The period is proven to be invariant under

- Completion/decompletion
- Planar duality for $G$
- Schnetz twist

\[ \text{The } c_2 \text{ invariant should have these symmetries as well.} \]
Expanded Laplacian and more polynomials

We need some definitions to obtain our combinatorial rephrasings. Let

\[ M_G = \begin{bmatrix} \Lambda & E^T \\ -E & 0 \end{bmatrix} \]

where \( \Lambda = \text{diag}(a_1, a_2, \ldots, a_n) \) and \( E \) is the signed incidence matrix with one row removed.

Then as another way to view the matrix tree theorem we have

\[ \Psi_G = \det M_G \]

We also care about minors

\[ \psi^I_{G, K} = \det M_G(I, J)|_{a_e=0, e \in K} \]
Brown's integration algorithm is controlled by the denominators. These are polynomials with combinatorial meaning.

Let's revisit the sketch of the algorithm.
Spanning forest polynomials

These polynomials can all be rewritten as sums over spanning forests.

(Eg if edges 1, 2, 3 meet at a 3-valent vertex:

\[ \Psi_{1,2}^3 = \text{spanning heads} \]

\[ \Psi_{13,23} = \text{colours say vertices of the same colour are in the same tree of the spanning forest} \]
A result

Brown and Schnetz conjecture that for all $p$, 4-regular $K$, $v_1, v_2 \in V(K)$

$$c_2^{(p)}(K - v_1) = c_2^{(p)}(K - v_2)$$

I prove that if $K$ has an odd number of vertices, $v_1, v_2 \in V(K)$, then

$$c_2^{(2)}(K - v_1) = c_2^{(2)}(K - v_2)$$
Two known results we need

Proposition (Brown and Schnetz)

\[ c_2^{(p)}(G) = [\psi_{G,3}^{1,2} \psi_{G}^{13,23}]_p \mod p \]

Proposition (Corollary of Chevalley-Warning)

If \( f \) has total degree \( n \) in \( x_1, x_2, \ldots, x_n \) then

\[ [f]_p = \text{coefficient of } x_1^{p-1} \cdots x_n^{p-1} \text{ in } f^{p-1} \mod p \]
Reduction to counting certain edge bipartitions

Apply these to our situation.

\[ c_2^{(2)}(G) = \left[ \psi_{G,3}^{1,2} \psi_{G}^{13,23} \right]_2 \mod 2 \]

\[ = \text{coefficient of } x_1 \cdots x_n \text{ in } \psi_{G,3}^{1,2} \psi_{G}^{13,23} \mod 2 \]

\[ = \# \text{ of bipartition of the edge} \]

\[ \vdots \text{giving a sparsely tree of } G \cdot 123 \]

\[ \vdots \text{determining a forest compatible with} \]

\[ \vdots \text{mod} \]
Proof sketch

\[ \# S(2l-3) \]
Numbers in QFT  Algebro-geometric objects  Combinatorial rephrasings