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A priori estimates for Yang-Mills Fields

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The Yang-Mills equations, introduced by theoretical physicists into mathematics, continue to yield a fascinating supply of new forms, facets and applications [A-B], [D1], [D2], [Si]. Technically as partial differential equations Yang-Mills is fairly well understood in dimension four and below. In dimension $n > 4$, previous techniques apply only if a local bound on the $L^{n/2}$ norm of the curvature is known. This is not a natural condition. This paper presents the rather technical result that, given any smooth solution of a Yang-Mills type equation (including a lower order term), estimates on arbitrary L^p norms of the curvature, and hence higher derivatives, can be obtained from energy (L^2 bounds) by using the equation. This theory has content in dimension larger than four only. Similar results have been obtained for the equations by Nakajima [N]. I am indebted to Nakajima for pointing out an error in my first version of Chapter 2.

Although this type of result is standard in the theory of elliptic systems of non-linear partial differential equations, our work is special to Yang-Mills equations because of the gauge freedom. We rely heavily on previous work on gauge fixing. Otherwise the proofs are similar to and originate from the same ideas as Schoen's estimates for harmonic maps. The monotonicity or scaling formulas of section three extend the work of Price [P]. S.T. Yau and the author found these results useful in constructing Hermitian Yang-Mills fields in stable bundles on Kähler manifolds and some of our more complicated terms are included precisely to handle that case [U-Y]. I conjecture that estimates of this sort can be used to solve the Dirichlet problem for domains of dimension larger than 4 with singular sets of Hausdorf dimension at most $n - 4$.

In section one we formulate the usual Yang-Mills equations, describe previously understood gauge fixing techniques and regularity theory and give our precise result. In section two the holomorphic (or Hermitian) Yang-Mills theory is described and our results derived for this special case. In section three we prove both monotonicity formulas, which allow normalized L^2 estimates on balls to pass down to smaller balls, for the ordinary Yang-Mills theory and the holomorphic theory.

The main interest in the estimates as they stand is a weak convergence theory. Given the results of this paper, a sequence of "approximate" Yang-Mills fields with bounded energy has a subsequence which converges off a set of finite Hausdorff dimension $\dim M - 4$. The corresponding theorem for harmonic maps says the singular set is of dimension $\dim M - 2$. This argument can be found in [Se], [N] and [U-Y] and we do not repeat it. This theory should be part of a more general theory encompassing both the theory of harmonic maps and Yang-Mills. The necessary ingredients are monotonicity and local regularity estimates. Gauge theory differs from harmonic map theory in the necessity for choosing local gauges as well as in the lack of a good notion of a weak (L_1^2) solution. In any case, past experience has shown that general abstract theories are not good approaches to non-linear partial differential equations. A more interesting challenge is to find a third set of geometric equations with properties similar to the harmonic map and Yang-Mills equations.

1. The Yang-Mills Equation for a Riemannian Manifold.

The Yang-Mills equations are formulated as equations for a connection in a vector bundle E over a Riemannian manifold M. For simplicity we assume the gauge group is $U(N)$ or a subgroup of $U(N)$. It must in any case be compact. The metric in $U(N)$ we take to be the bi-invariant trace inner product

$$(V,W) = \operatorname{tr} VW^* = -\operatorname{tr} VW$$

for $V, W \in u(N)$. The independent variables are encoded in a connection D on E which is locally writen in a trivialization over an open set $\Omega \subset X$ as

(1)
$$E \mid \Omega \simeq \Omega \times \mathbb{C}^N$$
$$D = d + A \quad \text{where} \quad A(x) = \sum_j A_j(x)\, dx\ .$$

Here $A_j(x)$ is a skew-Hermitian $N \times N$ complex matrix. Local gauge transformations are maps $s : \Omega \to U(N)$ and the change in the potential is given by

(2)
$$\hat{A} = s^{-1} A s + s^{-1}\, ds$$
$$\hat{A}_i(x) = s^{-1}(x)\left(A_i(x) s(x) + \frac{\partial s}{\partial x^i}(x)\right)\ .$$

There are no a priori estimates possible for A, since even $A = 0$ can be gauge-charged into a singular $\hat{A} = s^{-1}\, ds$ by choosing a singular s. Invariant estimates are made on the field or curvature F. This is a $u(N)$ valued two-form

(3)
$$F = dA + \tfrac{1}{2}(A \wedge A) = \sum_{i,j} F_{ij}\, dx^i \wedge dx^j$$
$$F_{ij}(x) = \frac{\partial}{\partial x^i} A_j(x) - \frac{\partial}{\partial x^j} A_i(x) + [A_i(x)\,,\, A_j(x)]\ .$$

Curvature transforms by the *ad* action of s

(4)
$$\hat{F} = s^{-1} F s$$

which mean $|\hat{F}| = |F|$ as s is unitary. Therefore L^p estimates on curvature are invariant.

We already know that uniform L^p estimates on curvature for $\dim X < 2p$ imply the existence of a good gauge in which L^p norms on F lead to L_1^p estimates on good choices of A and weak compactness arguments. Our goal is to obtain the L^p estimates from the natural L^2 estimates and an equation. To write down the equation, we need a Riemannian metric, and the size of balls allowed in our theory depends in a crucial way on the geometry of this metric.

If $g = \sum_{i,j} g_{ij}\, dx^i dx^j$ is a Riemannian metric on X in local coordinates, then the full Yang-Mills equations are

$$D^*F = \sum_{i,j,t,\ell} \frac{\partial^A}{\partial x^i}(\det g^{-1} g^{ij} g^{\ell k} F_{jk}) g_{\ell t}\, dx^t = 0 \tag{6}$$

where $\frac{\partial^A}{\partial x^i} V = \frac{\partial}{\partial x^i} V + [A_i, V]$. We consider the equation

$$D^*F = Q \tag{7}$$

where $Q = \sum_\ell Q_\ell\, dx^\ell$ has a maximum estimate $\max_{x \in X} |Q(x)| \leq Q_\infty$ which is a priori known. Note this estimate is gauge invariant.

The Riemannian tensor g on M enters into the estimates in the following manner: all estimates are local and are carried out in Gaussian normal coordinates about an arbitrary point $\xi \in M$ in balls of radius at most σ_0 which are well inside the injectivity radius of M. We denote the geodesic balls by

$$B_\sigma(\xi) = \{\, x \in M : \delta(x, \xi) \leq \sigma \,\} \tag{8}$$

where $\sigma \leq \sigma_0$. The bound on σ_0 is determined by the following conditions

(a) $\sigma_0 <$ injectivity radius of (M, g).

(9) (b) $|g^{ij}(x) - \delta^{ij}| \leq K_0 |x|^2$

$$\left|\frac{\partial}{\partial x^k} g^{ij}(x)\right| \leq K_0 |x|$$

(c) $\sigma_0^2 K_0 \leq 1/2$

The constant K_0 depends on the Riemannian curvature of g.

The fundamental gauge fixing result can be interpreted on our Riemannian manifold in the context of these geodesic balls.

Theorem 1.1. *There exists $\kappa = \kappa(n, N)$ such that if $\sigma \leq \sigma_0$ and*

$$\int_{B_\sigma(\xi)} |F|^{n/2} * 1 \leq \kappa^{n/2}$$

then there exists a unique choice of gauge (up to constant rotations of the fiber) in which $d^*A = 0$ on $B_\sigma(\xi)$, $A \mid \text{normal} = 0$, and

$$\int_{B_\sigma(\xi)} |\nabla A|^{n/2} * 1 + \left(\int_{B_\sigma(\xi)} |A|^n * 1 \right)^{1/2} \leq C \int_{B_\sigma(\xi)} |F|^{n/2} * 1 \, .$$

Here $C = C(n, N)$.

Proof. For the flat metric this is the main theorem in the first part of an earlier paper **[U-1]**. The condition $\sigma \leq \sigma_0$ shows that the geometry of $B_\sigma(\xi)$ is close to that of the ball of radius ξ in $\mathbb{R}^n$, so the same proof will work with a small but fixed modification of κ (to $\kappa/2$) and C (to $2C$). Technically in the proof we have solved $\sum_j \frac{\partial}{\partial x^j} A_j = 0$ rather than $d^*A = 0$ ($*$ in the g metric). A second application of the implicit function theorem to a slight modification of the original theorem allow us to solve $d^*A = 0$.

As a fairly straightforward corollary, we obtain L^p estimates for any p. We find it convenient to introduce the notation

$$\left(\sigma^{2p-n} \int_{B_\sigma(\xi)} |F|^p * 1 \right)^{1/p} = f_p(\xi, \sigma) (= f_p) \, . \tag{10}$$

This is a scale invariant quantity, in that we can always rescale our original manifold (and K_0) to work in the ball $\sigma = 1$. This is explained a little less briefly in chapter 3.

Theorem 1.2. *Let $D^*F = Q$, $|Q(x)| \leq Q_\infty$ and $(\int_{B_\sigma(\xi)} |F|^{n/2} * 1)^{2/n} = f_{n/2}(\xi, \sigma)$. Then there exists a constant $\kappa_{n/2}$ such that for $\sigma \leq \sigma_0$, $2 \leq p < n$, $f_{n/2}(\xi, \sigma) \leq \kappa_{n/2}$ and some c_p*

$$\begin{aligned} \left(\sigma^{2p-n} \int_{B_{\sigma/2}(\xi)} |F|^p * 1 \right)^{1/p} &= f_p(\xi, \sigma/2) \\ &\leq c_p (f_2(\xi, \sigma) + \sigma Q_\infty) \, . \end{aligned}$$

Proof. We rescale to prove in the unit ball with a metric satisfying (9) for $\sigma_0 = 1$. We use the gauge of Theorem 1.2. Our first step is to point out

$$\|A\|_{1,2} \le c_1 \|F\|_{0,2} = c_1 f_2 \tag{11}$$

as shown in Lemma 2.5 of [U-1]. Now write the two equations $d^* A = 0$ and $D^* F = 0$ together in a single Laplace-type equation for A.

$$\Delta_g A + [A, dA] + [A,[A,A]] = Q\ . \tag{12}$$

Here Δ_g is the Laplace operator $d^*d + dd^*$ on one-forms and the exact form of the non-linearities is not important, only the type.

In the range chosen for p, the multiplication theorems

$$L_1^p \otimes L^{n/2} \to L_{-1}^p$$

are valid, although as $p \to n$ the norm $\tilde{c}_p$ of the estimate becomes infinite. Regard equation (12) as a linear equation for $A = \phi$ with lower order coefficients $dA \in L^{n/2}$, $[A,A] \in L^{n/2}$

$$L_A(\phi) = (\Delta_g - 1)\Phi + [\Phi, dA] + [\phi,[A,A]] = Q - A\ .$$

The linear operator is a perturbation of size $\tilde{c}_p f_{n/2}$ from $\Delta_g - 1$ as a map $L_1^p \to L_{-1}^p$. As such, both the uniqueness and interior estimates hold if $\tilde{c}_p f_{n/2}$ and $\sigma_0^2 K$ are sufficiently small. In particular we get an interior estimate on $B_{1/2}$,

$$\|\phi\|_{1,p,1/2} \le c_p'(\|\phi\|_{1,2,1} + \|L_A\phi\|_{-1,p,1})\ .$$

For $\phi = A$ this gives

$$\|A\|_{1,p,1/2} \le c_p'(\|A\|_{1,2} + Q_\infty)\ . \tag{13}$$

Equations (12) and (13) transform into the gauge invariant estimate on curvature

$$f_p(0,1/2) \le c_p(f_2(0,1) + Q_\infty)\ .$$

This is stated in the scaled result as the theorem. This result is quite similar to results in [U-1].

Further improvement is standard. We rewrite (12) as

$$\Delta_g A = Q + [A, dA] + [A, [A, A]] \ . \tag{15}$$

Estimates on Q plus an initial L_1^p estimate on A yield interior estimates on arbitrary higher Sobolev norms of A using the iteration of the standard L^q estimates for the Laplace operator. We state one corollary in its gauge invariant form. It is proved by applying the theorem to estimate $f_p(\xi, \sigma/2)$ and then iterating the estimate again.

Corollary 1.3. *Let $f_{n/2}(\xi,\sigma) \le \kappa_{n/2}$ and $f_2(\xi,\sigma) \le \varepsilon$ for $n/2 < p$, $\sigma \le \sigma_0$. Then for $x \in B_{\sigma/4}(\xi)$*

$$\sigma^2 |F(x)| \le K_n(1 + \varepsilon + Q_\infty \sigma)(\varepsilon + Q_\infty \sigma)$$

We now use the monotonicity on scaling inequalities as proved in Theorem 3.2. For convenience we state the inequality here. For $K_0\sigma^2 \le 1/2$ as described in (9) we have from Theorem 3.2

$$f_2^2(\xi,\sigma) \le w_0\big(f_2^2(\xi,\rho) + Q_\infty^2 \rho^2\big) \ . \tag{16}$$

Here $w_0 = \exp w\sigma_0^2(w(n)K_0 + 1)\sigma_0^2$ where $w(n) \sim n!$ is a combinatorial constant depending on $n = \dim M$.

Lemma 1.4. *There exists $\varepsilon_0 > 0$ such that if $4\rho < \sigma_0$, $f_2(\xi, 4\rho) = \varepsilon < \varepsilon_0$, $Q_\infty \rho < \varepsilon$, then for all $\sigma < \rho$, $x \in B_{3\rho}(\xi)$*

$$f_2(x,\rho) \le w_1 \varepsilon \ .$$

where $w_1^2 = w_0(4^{n-4} + 1)$.

Proof. From (16) we obtain

$$f_2^2(x,\sigma) \le w_0\big(f_2^2(x,\rho) + Q_\infty^2 \rho^2\big) \ .$$

However $B_\rho(x) \subset B_{4\rho}(\xi)$ and we may conclude $f_2^2(x,\rho) \le 4^{n-4} f_2^2(\xi, 4\rho)$. This gives our inequality immediately.

Lemma 1.5. *Assume the hypotheses of Lemma 1.4, and require in addition that* $f_{n/2}(x,\sigma) \le \kappa_{n/2}$ *for some x and σ. Then*

$$f_{n/2}(x, \sigma/2) \le \overline{C}_{n/2}\varepsilon \,.$$

$$\sigma^2|F(y)| \le \overline{K}_n\varepsilon \quad \text{for} \quad y \in B_{\sigma/4}(x) \,.$$

Proof. First apply Theorem 1.2 with $p = n/2$. Note $\overline{C}_{n/2} = C_{n/2}(w_1+1)$. Now apply Corollary 1.3. We can compute exactly that

$$\overline{K}_n = K_n\big(1 + (w_1+1)\varepsilon_0\big)(w_1+1) \,.$$

We are now set up to prove rather easily our main theorem. On a compact manifold with $\int_M |F|^2 \le \varepsilon_0^2$, the argument is very easy. One finds $\max_{x\in M}|F(x)| = |F(y)|$, chooses if possible a ball of the right size in which

$$\left(\int_{B_\sigma(y)} |F(x)|^{n/2} * 1\right)^{2/n} = f_{n/2}(y,\sigma) = \kappa_{n/2} \,,$$

and obtains a max estimate in $B_{\sigma/4}(y)$ from Lemma 1.5. Since $|F(x)| \le |F(y)|$ we get an estimate in all of M. If ε_0 is sufficiently small, we obtain a contradiction. The elaboration in the proof of Theorem 1.6 comes from the need to adjust the size of balls depending on their distance from the boundary of the large ball in question. This adjustment could be accomplished in a number of ways.

Theorem 1.6. *Let $D^*F = Q$ with $\max_{x\in M}|Q(x)| \le Q_\infty$. There exists $\varepsilon_0 > 0$ such that if $4\rho < \sigma_0$, $Q_\infty\rho < \varepsilon \le \varepsilon_0$ and $f_2(\xi, 4\rho) < \varepsilon \le \varepsilon_0$, then*

$$\max_{x\in B_\rho(\xi)} \rho^2|F(x)| \le \overline{K}_n\varepsilon \,.$$

Proof. Define a function

$$q : B_{3\sigma} \times [0,1] \to \mathbb{R}^+$$

by the formulas

$$q(x,t) = f_{n/2}\big(x, t\omega(x)\big)$$

$$\omega(x) = \begin{cases} (3\rho - \delta(x,\xi))/2 & \rho \le \delta(x,\xi) \le 3\rho \\ \rho & \delta(x,\xi) \le \rho \end{cases} \tag{17}$$

By constructing the function $\omega(x) = \omega(x, \xi, \rho)$ is continuous. Moreover, $q(x,t) = 0$ for $t = 0$ and for $x \in \partial(B_{3\sigma}(\xi))$. The heart of the proof is to show $q(x,t) \leq \kappa_{n/2}$ in the entire domain $(x,t) \in B_{3\rho}(\xi) \times [0,1]$. Once we have done this, we let $t = 1$ and $x \in B_\rho(\xi)$ to get

$$f_{n/2}(x,\rho) = q(x,1) < \kappa_{n/2} \, .$$

Now Lemma 1.5 yields the estimate

$$\rho^2 |F(x)| \leq \overline{K}_n \varepsilon$$

by taking the center x of each ball $B_\rho(x)$ as y.

To show $q(x,t) < \kappa_{n/2}$ for all (x,t) in the domain and for ε_0 sufficiently small, assume this is not true. Due to continuity and $q(x,0) = 0$, we can assume that there exists a $0 < \tau \leq 1$ and a $y \in B_{3\sigma}(\xi)$ with the properties

$$\begin{aligned} &q(y,\tau) = \kappa_{n/2} \\ &q(x,t) \leq \kappa_{n/2} \quad \text{for} \quad t \leq \tau \, . \end{aligned}$$

Moreover, y is strictly in the interior of $B_{3\sigma}(\xi)$ since $q(x,t) = 0$ for $x \in \partial B_{3\sigma}(\xi)$.

The point of the argument is that we are still able to apply Lemma 1.5, since $q(x,\tau) = f_{n/2}(x, \tau\omega(x)) \leq \kappa_{n/2}$. This gives us the estimate in the ball of half the radius

$$f_{n/2}(x, \tau/2\omega(x)) \leq \overline{C}_{n/2}\varepsilon \, . \tag{18}$$

On the other hand, the integral $f_{n/2}^{n/2}(y, \tau\omega(y))$ can clearly be estimated in terms of a sum of integrals over the smaller balls $\sum_i f_{n/2}^{n/2}(x_i, \tau\omega(x_i))$ for $x_i \in B_{\tau\omega(y)}(y)$. Alternately, we try a double integral and change the order of integration. By construction

$$\kappa_{n/2}^{n/2} = f_{n/2}^{n/2}(y, \tau\omega(y)) = \int_{B_{\tau\omega(y)}(y)} |F(y)|^{n/2} + 1 \, .$$

However,

$$
\begin{aligned}
&\int_{x\in B_{\tau\omega(y)}(y)} \left(\int_{\eta\in B_{\tau/2\,\omega(x)}(x)} |F(\eta)|^{n/2} * 1 \right) * 1 \\
(19) \qquad &= \int_{\eta\in B_{s}(y)} \left(\int_{x\in B_{\tau\omega(y)}(y)\cap B_{\tau/2\,\omega(x)}(\eta)} *1 \right) |F(\eta)|^{n/2} * 1 \\
&\geq m \int_{\eta\in B_{\tau\omega(y)}(y)} |F(\eta)|^{n/2} * 1 = m\kappa_{n/2}^{n/2} .
\end{aligned}
$$

Here the number m can be estimated exactly as

$$
m = \min_{\eta\in B_{\tau\omega(y)}(y)} \left(\int_{x\in B_{\tau\omega(y)}(y)\cap B_{\tau/2\,\omega(x)}(\eta)} *1 \right)
$$

- From the definition of $\omega(x)$, if $\delta(x,y) \leq \omega(y)$ we can conclude $\omega(x) \leq 1/2\omega(y)$. So it suffices to estimate a lower bound for the area of $B_{\tau\omega(y)}(y) \cap B_{\tau/4\,\omega(y)}(\eta)$ when $\eta \in B_{\tau\omega(y)}(y)$. Clearly the worst case is when $\eta \in \partial B_{\tau\omega(y)}(y)$ when slightly under half the ball $B_{\tau/4\,\omega(y)}(\eta)$ lies in $B_{\tau\omega(y)}(y)$. In any case

$$
m \geq m(n)\big(\tau\omega(y)\big)^{n} .
$$

If we substitute this inequality in (19) and use (18) we have

$$
\begin{aligned}
m(n)\big(\tau\omega(y)\big)^{n}\kappa_{n/2}^{n/2} &= \int_{x\in B_{\tau\omega(y)}(y)} f_{n/2}^{n/2}\big(x, \tau/2\,\omega(x)\big) * 1 \\
&\leq (\overline{C}_{n/2}\varepsilon)^{n/2} \ \operatorname{vol} B_{\tau\omega(y)}(y) \\
&\leq (\overline{C}_{n/2}\varepsilon)^{n/2} V_n\big(\tau\omega(y)\big)^{n} .
\end{aligned}
$$

This is impossible if ε is too small relative to $\kappa_{n/2}$. This completes the proof.

2. The Holomorphic Yang-Mills Equations and the Hermitian-Einstein Tensor

The holomorphic Yang-Mills equations are formulated in a vector bundle E over a complex Kähler manifold X, with the additional assumption that the bundle E has a holomorphic structure. The connections we allow are to be compatible with this holomorphic structure. The existence of the holomorphic structure means precisely that there is a holomorphic trivialization

$$
\begin{aligned}
&E \mid \mathcal{O} \cong \mathcal{O} \times \mathbb{C}^N \\
(20) \qquad &D = d + A \\
&A(x) = \sum_\alpha A_\alpha(x)\, dz^\alpha
\end{aligned}
$$

where $A_\alpha(x) \in \mathrm{GL}(N, \mathbb{C})$ and $A_{\bar\alpha}(x) = 0$. In other words, in holomorphic coordinates the covariant derivative agrees with ordinary differentiation in the $z^{\bar\alpha} = x^\alpha - iy^\alpha$ directions. Because changes in the holomorphic coordinates in the bundle are holomorphic maps $s : \mathcal{O} \to \mathrm{GL}(N, \mathbb{C})$, this is a coordinate invariant definition of a compatible connection.

Since $\bar D = \bar\partial$ in holomorphic coordinates, it follows that the curvature $F_{\bar\alpha,\bar\beta} = 0$. The converse is also true. By the Newlander-Nirenberg theorem, if a connection with $F_{\bar\alpha,\bar\beta} = 0$ is given, this connection defines a holomorphic structure on E. In our estimates we make use of the fact that the information about the holomorphic structure on E is encoded in any compatible connection. For our global results, there is no need to fix the holomorphic structure.

Our results apply only to bundles with connections which are compatible with some holomorphic structure and whose structure group reduces to a compact $U(N)$ group. The results are certainly not true for non-compact groups. Given a holomorphic bundle E, and $\bar D = \bar\partial$ part of a connection, a unitary connection is equivalent to the choice of a metric h on E. In holomorphic coordinates

$$(21) \qquad D = (\bar\partial,\ \partial + h^{-1}\partial h)\,.$$

Note that if $h = u^* u$ locally, then the gauge equivalent

$$\hat D = u \circ D \circ u^{-1}$$

has the local form

$$\hat{D} = \left(\bar{\partial} + u\,\bar{\partial}(u^{-1})\,,\ \partial + (u^*)^{-1}\partial u^*\right) . \tag{22}$$

In this unitary and *not holomorphic* choice of frame, the connection form has the formula

$$A_{\bar{\alpha}} = \frac{\partial}{\partial z^{\bar{\alpha}}}(u)\,u^{-1}$$

$$A_\alpha = (u^*)^{-1}\frac{\partial}{\partial z^\alpha}(u^*) = -A^*_{\bar{\alpha}} .$$

In real coordinates these connection forms are skew-Hermitian just as in the first chapter on real manifolds.

We have not yet explained where the Kähler metric will come in. Suppose X has a Kähler metric with Kähler form $\omega \in \Omega^{1,1}(X)$. For any unitary connection compatible with any complex structure on E, as explained before $F \in \Omega^{1,1}(X, AdE)$. The Hermitian-Einstein tensor is by definition

$$H = (\omega, F) - \mu I \in C^\infty(AdE) . \tag{23}$$

The $\mu \in \mathbb{R}$ is a normalization constant depending on ω and $c_1(E)$ which is chosen for the purpose of global calculations on compact manifolds in such a way to insure

$$\int_X \operatorname{tr} H\omega^m = 0 .$$

Our calculations are local and we assume $\mu = 0$. The relationship with Yang-Mills becomes apparent given the following theorem.

Theorem 2.1. (Kobayashi [K]) *If D is a connection compatible with any holomorphic structure on E, then $(\omega, F) = \mu I$ implies the Yang-Mills equations.*

Proof. Since $\mu \in \mathbb{R}$, $D(\omega, F) = 0$. However, one can compute that

$$D(\omega, F) = (\omega, DF) + (D^*F)_\gamma\, dz^\gamma - (D^*F)_{\bar{\gamma}}\, dz^{\bar{\gamma}} .$$

By the Bianchi identities $DF = 0$, and for connections with curvatures in $\Omega^{1,1}(AdE)$, $D^*F = 0$ is equivalent to $D(\omega, F) = 0$.

In the technical process of constructing solutions to the equation $H = 0$ on a compact Kähler manifold, it becomes useful to examine the inequality $\|H\|_{L^\infty} \leq B$. The local form of these equations is considerably simpler than the full Yang-Mills equations

$$\sum_{\bar{\alpha},\beta} g^{\bar{\alpha}\beta} F_{\bar{\alpha},\beta} = H \tag{24}$$

$$F_{\bar{\alpha},\bar{\beta}} = F_{\alpha,\beta} = 0 \,.$$

The results and the method of obtaining them are only slightly different in the holomorphic case. To start with, we need similar estimates on the geometry.

The Kähler form on X enters into the estimates in much the same way the Riemannian tensor did in Chapter I. All estimates are done locally in holomorpic coordinates about an arbitrary point $\xi \in X$, in balls of radius at most σ_0, well inside the injectivity radius. We assume that we work in a region of X in which every point has such a holomorphic chart $O_\xi \simeq \{(z^1, \ldots, z^m) : |z| \leq \sigma_0\}$, which depends continuously on $\xi \in X$. There may be global topological obstructions, but our theory is all local. Let $B_\sigma(\xi) = \{z \in O_\xi : |z| \leq \sigma\}$. The bound on σ_0 and the geometry are determined by the following restrictions on the Kähler form in these coordinates

(25)

(a) $B_{\sigma_0}(z) \subseteq X$ is a holomorphic coordinate.

(b) $|g^{\alpha,\bar{\beta}}(z,\bar{z}) - \delta^{\alpha,\bar{\beta}}| \leq K|z|^2$

$$\left|\frac{\partial}{\partial z^\gamma} g^{\alpha,\bar{\beta}}(z,\bar{z})\right| \leq K|z|$$

In repeating ourselves from the Riemannian case, Theorem 1.1 applies as is to the holomorphic case. The definition for complex dimension $m = n/2$

$$f_p(\xi,\sigma) = \left(\sigma^{2(p-m)} \int_{B_\sigma(x)} |F|^p * 1\right)^{1/p}$$

still applies. Since we assume the holomorphic coordinates centered at ξ depend continuously on ξ as we move ξ throughout our ball (remember we don't have Gaussian normal coordinates any more), we still have that $f_p(\xi,\sigma)$ is continuous in $\xi \in X$. The analog of Theorem 1.2 is the following:

Theorem 2.2. *Let* $(\omega, F) = H$, $|H(x)| \leq H_\infty$ *and*

$$\left(\int_{B_\sigma(x)} F^m * 1 \right) = f_m(\xi, \sigma) .$$

Then there exists a constant κ'_m *such that for* $\sigma \leq \sigma_0$, $2 < p < 2m$ *and* $f_m(\xi,\sigma) < K'_m$

$$\left(\sigma^{2(p-m)} \int_{B_{\sigma/2}(\xi)} |F|^p * 1 \right)^{1/p} \leq c'_p (f_2(\xi,\sigma) + \sigma^2 H_\infty) .$$

Note the simplification in terms of the complex dimension $m = n/2$.

Proof. As in the Riemannian case, we rescale to the unit ball in the holomorphic coordinate chart. Note that Q scaled like a one-form, but H now scales like a two form, which explains the $\sigma^2 H_0$ from scaling by σ. We obtain as the first step the result of Theorem 1.1.

$$d * A = \sum g^{\bar{\alpha},\beta} \left(\frac{\partial}{\partial z^{\bar{\alpha}}} A_\beta + \frac{\partial}{\partial z^\beta} A_{\bar{\alpha}} \right) = 0 \tag{26}$$

$$\|A\|_{1,2} \leq c_1 \|F\|_{0,2} = c_1 f_2$$

in the unit ball. Our equations are the following over-determined elliptic system for the real form $\sum_\alpha A_\alpha dz^\alpha + A_{\bar{\alpha}} dz^{\bar{\alpha}}$ with $A_\alpha = -A^*_{\bar{\alpha}}$.

$$\begin{aligned} &d * A = 0 \\ &\left(\omega \cdot (dA + [A, A]) \right) = H \\ &(dA + [A, A])^{(0,2)} = 0 . \end{aligned} \tag{27}$$

To see the form of this system, it can be written for $\sum_\alpha A_\alpha dz^\alpha$ as

$$\begin{aligned} g^{\bar{\alpha},\beta} \frac{\partial}{\partial z^{\bar{\alpha}}} A_\beta + \tfrac{1}{2}[A_{\bar{\alpha}}, A_\beta] &= \frac{1}{2} H . \\ \frac{\partial}{\partial z^\alpha} A_\beta - \frac{\partial}{\partial z^\beta} A_\alpha + [A_\alpha, A_\beta] &= \frac{1}{2} H . \end{aligned} \tag{27'}$$

View $A_{\bar{\alpha}}$ and one factor of the quadratic term as coefficients and we have an over-determined elliptic linear system $L_A \phi = H$ for $A = \phi$. In the range we are working

$$L^n \otimes L^p_1 \to L^p .$$

by multiplication. From Theorem 1.1

$$\|A\|_{0,n} \leq c_m^m f_m \leq c_m^m \kappa'_m .$$

So the linear operator L_A is a perturbation of size $c'_p f_m$ for

$$L\Phi = \{ \ (\omega, \bar{\partial}\Phi) \ , \ \partial\Phi \ \} .$$

Since the geometry is nearly flat, we are in fact near a set of equations which are familiar from analysis of the Dolbeault complex.

$$\sum_\alpha \frac{\partial}{\partial z^{\bar{\alpha}}} A_\alpha = -\tfrac{1}{2} H$$

$$\frac{\partial}{\partial z^\alpha} A_\beta - \frac{\partial}{\partial z^\beta} A_\alpha = 0 .$$

Under the conditions that the coefficients which perturb the system from this familiar linear system are small, we obtain

$$\|\phi\|_{1,p,B_{1/2}} \leq c'_p \big(\|\phi\|_{1,2,B_1} + \|L_A \phi\|_{0,p,B_1} \big) .$$

This smallness criterion is met by Theorem 1.2 when $f_m(\xi, \sigma) \leq \kappa'_m$. Remember that $m = n/2$. The theorem follows by letting $\phi = A$ and estimating $\|H\|_{L^p}$ by H_∞.

The rest of the proof essentially follows the Riemannian case with one not very essential difference. In the Riemannian case it seemed most natural to obtain maximum estimates on the curvature tensor F, although under the given hypothesis we certainly could have obtained L^p estimates for all p on the covariant derivative of F. In the complex case, we assume a maximum estimate on H, which is one order less than a maximum estimate on Q. Since elliptic systems do not invert from L^∞ to L_1^∞, we cannot expect to use (26) and the maximum estimates on H to obtain maximum estimates on F. The equivalent of the Riemannian theorem would require maximum estimates on $DH \simeq Q$. This is too strong for certain applications which we have in mind. We expect to be able to obtain L^p estimates on F for all p.

Hence for the rest of this chapter we pick a fixed Sobolev exponent ℓ, $2m < \ell < \infty$. Replace the scale invariant $\sigma^2 F(x)$ of the Riemannian case by the scale invariant $f_\ell(x,\sigma)$. Recall that

$$f_\ell(x,\sigma) = \Big(\sigma^{2(\ell-m)} \int_{B_\sigma(x)} |F|^\ell * 1\Big)^{\frac{1}{\ell}} .$$

The complex equivalent of Corollary 1.3 is the following.

Corollary 2.3. *Assume the hypotheses of Theorem 2.2 are valid. Then*

$$f_\ell\left(\xi, \frac{\sigma}{4}\right) \leq K'_m(1+\varepsilon+\sigma^2 H_\infty)(\varepsilon+\sigma^2 H_\infty) .$$

Proof. We apply Theorem 2.2. This gives us an estimate in $B_{\frac{\sigma}{2}}(\xi)$. Take the elliptic system (27) and put the non-linearities on the right-hand side.

$$d * A = 0$$

$$(\omega \cdot dA) = -(\omega, [A, A])$$

$$[dA]^{0,2} = -[A, A]^{0,2} .$$

The right-hand side is in L^q for $q = \frac{p}{2} < m$. Hence by elliptic regularity theory, $A \in L^q_1(\Omega)$ for $\bar{\Omega} \subset$ interior of $B_{\frac{\sigma}{2}}(\xi)$. By the Sobolev embedding theorem, $A \in L^{q'}$ for $q' = \frac{qm}{(m-q)}$. Repeat the interior elliptic regularity estimate using q' instead of q. By judicious choice of $p < 2m$, $q' = \ell$ for our chosen ℓ. Our estimate keeps track of rescaling for the small balls we work in.

We now borrow again from section 3 ahead and use the monotonicity formulas for the complex case. From Theorem 3.5 we obtain in a straightforward way the next inequality.

Lemma 2.4. *There exists $\varepsilon'_0 > 0$ such that if $4\rho < \sigma_0$, $f_2(\xi,\rho) = \varepsilon < \varepsilon'_0$ and $H_\infty \rho^2 < \varepsilon$, then for $\sigma < \rho$ and $x \in B_{2\rho}(\xi)$*

$$f_2(x,\rho) \leq w'_1 \varepsilon$$

where $w'_1 = (1+K\sigma_0^2)^{m-3} \max(1+K\sigma_0,\ v_{2m})$.

In the final preparatory lemma, we again copy Lemma 1.5, but with f_ℓ replacing the maximum estimate.

Lemma 2.5. *Suppose the hypotheses of (2.3) and (2.4) are valid. Then*

$$f_m\left(x, \frac{\sigma}{2}\right) \leq C'_m \varepsilon$$
$$f_\ell(y) \leq K'_m \varepsilon$$

for $y \in B_{\frac{\sigma}{2}}(x)$ and $\rho \leq \frac{1}{2}(\frac{\sigma}{2} - |y - x|)$.

Proof. First apply Theorem 2.2 with $p = m$. This gives

$$f_m\left(\xi, \frac{\sigma}{2}\right) \leq C'_m\left(\varepsilon + \sigma^2 H_\infty\right) \leq 2C'_m \varepsilon .$$

The quantity $f_m(\xi, \sigma)$ needs no renormalization factors, so

$$f_m(y, 2\rho) \leq f_m\left(\xi, \frac{\sigma}{2}\right)$$

for $B_{2\sigma}(y) \subset B_{\frac{\sigma}{2}}(\xi)$. Now apply Corollary 2.3 with $\xi = y$ and σ replaced by ρ to get the Lemma.

Theorem 2.6. *Let $(\omega, F) = H$ and assume $\max_{x \in M} |H(x)| \leq H_\infty$. There exists $\varepsilon' > 0$ such that if $4\rho < \sigma_0$, $H_\infty \rho^2 < \varepsilon \leq \varepsilon'$ and $f_2(\xi, 4\rho) < \varepsilon \leq \varepsilon'$, then*

$$f_\ell(x, \rho) = \left(\sigma^{2(\ell - m)} \int_{B_\rho(x)} |F(\xi)|^\ell * 1\right)^{\frac{1}{\ell}} \leq \bar{K}'_m \varepsilon .$$

Proof. The proof of Theorem 1.6 carries over precisely to this case. We define $q(x, t)$ in the same way, use (2.2)–(2.4) instead of (1.2)–(1.4), and primed quantities instead of unprimed. We then get

$$f_{\frac{n}{2}}(x, \rho) = q(x, 1) < K'_{\frac{n}{2}} .$$

An application of Lemma 2.5 yields

$$f_\ell(x, \rho) \leq \bar{K}'_n \varepsilon .$$

3. The Scaling Inequalities

The scaling or monotonicity formulas transfer normalized L^2 estimates in a ball of arbitrary radius down to normalized estimates on smaller balls. These formulas play a central role in the regularity theory of harmonic maps [S], [S-U] and we expect them to be as important in the general theory of Yang-Mills equations. The original monotonicity formula for solutions of Yang-Mills equations is due to Price [P]. We prove two different formulas. The first is for solutions of $D^*F = Q$ on Riemannian manifolds and the second for solutions of $(\omega, F) = H$ which are holomorphic connections on Kähler manifolds.

In the theory of harmonic maps $s : M \to N$, the quantity ds is a (tensor-valued) one-form, the scale invariant quantities are the integrals

$$\sigma^{-n+2} \int_{B_\sigma(\xi)} |ds|^2 * 1 = s(\xi, \sigma)$$

and the monotonicity formulas for harmonic maps take the form

$$s(\xi, \sigma) \le \hat{c}s(\xi, \rho)$$

for $\sigma \le \rho$ [S], [S-U]. In Yang-Mills theory, the connection form A is a one-form and the energy density is constructed from F_A, which scales as a two-form.

$$F_A = F_{ij}\, dx^i \wedge dx^j\,.$$

The correctly scaled integrals, the counterparts of the $s(\xi, \sigma)$, have already been introduced in (10)

$$\sigma^{4-n} \int_{B_\sigma(\xi)} |F_A|^2 * 1 = f_2^2(\xi, \sigma)\,.$$

Keep in mind that the results we wish to prove come from simple integration by parts formulas in $\mathbb{R}^n$ and $\mathbb{C}^m$ on Yang-Mills connections. The complications in our theory derive solely from the incorporation of the curvature of M and the lower order terms in the estimates.

We start with the Riemannian case. Let

25) $$F^2 = \sum g^{\ell k} \operatorname{tr} F_{i\ell} F^*_{jk}\, dx^i \otimes dx^j$$

be the symmetric two tensor on M obtained by contracting $\operatorname{tr} F \otimes F^*$ once with the metric. Contracting twice will clearly give $|F|^2$. In our application of the calculation in the next lemma, $\varphi(x) = \frac{1}{2}\delta(x,\xi) \simeq \frac{1}{2}|x-\xi|^2$ and $d\varphi(x) \simeq x$. The inner products and divergence are all taken with respect to the Riemannian metric.

Lemma 3.1. *Let $D^*F = Q$ and φ be a real-valued function on M. Then*

$$d^*\left((d\varphi, F^2) - \tfrac{1}{4}d\varphi(g, F^2)\right) = (W, F^2) - \operatorname{tr}(d\varphi \otimes Q, F)\,.$$

Here W is the symmetric two-tensor $W = d^2\varphi - \frac{1}{4}g\Delta\varphi$.

Proof. Calculate at a single, arbitrary point $x \in M$ in Gaussian normal coordinates in a gauge with $A(x) = 0$. Then by the Leibnitz rule, at x in coordinates

$$\begin{aligned}
&d^*\left((d\varphi, F^2) - \tfrac{1}{4}\Delta\varphi(g, F^2) - (W, F)\right)(x)\\
&= \sum_{k,j}\left[\frac{\partial}{\partial x^k}\left(\frac{\partial\varphi}{\partial x^j}F^2_{kj} - \frac{1}{4}\frac{\partial\varphi}{\partial x^k}F^2_{jj}\right)(x)\right.\\
&\qquad \left. - \left(\frac{\partial^2\varphi}{\partial x^k \partial x^j}(x) - \frac{1}{4}\sum_i\left(\frac{\partial}{\partial x^i}\right)^2\varphi(x)\delta_{kj}\right)F^2_{jk}(x)\right]\\
&= \sum_{k,j}\frac{\partial\varphi}{\partial x^j}(x)\left(\frac{\partial}{\partial x^k}F^2_{kj}(x) - \frac{1}{4}\frac{\partial}{\partial x^j}F^2_{kk}(x)\right)\\
&= \left(d\varphi, (d^*F^2 - \tfrac{1}{4}d|F|^2)\right)(x)\,.
\end{aligned}$$

Continue by evaluating the term in the right-hand side of the inner product above

$$\begin{aligned}
&(d^*F^2 - \tfrac{1}{4}d|F|^2)(x)\\
&= \operatorname{tr}\sum_{j,\ell}\left(\frac{\partial}{\partial x^j}F_{j\ell}(x)F^*_{\ell k}(x) + F_{j\ell}(x)\frac{\partial}{\partial x^j}F^*_{\ell k} - \frac{1}{2}F_{j\ell}\frac{\partial}{\partial x^k}F^*_{j\ell}\right)dx^k\\
&= \operatorname{tr}(Q, F^*) + \tfrac{1}{2}(F^*, DF) = \operatorname{tr}(Q, F)\,.
\end{aligned}$$

Here the extra terms $\sum(F_{j\ell}, \frac{\partial}{\partial x^j}F^*_{\ell k}) - \frac{1}{2}(F_{j\ell}, \frac{\partial}{\partial x^k}F^*_{j\ell})\, dx^k$ are rearranged using the anti-symmetry $F_{jk} = -F_{kj} = -F^*_{jk}$ to be written in terms of the three form $\sum \frac{\partial}{\partial x^k}F_{jk}\, dx^k \wedge dx^j \wedge dx^\ell$, which vanishes by the Bianchi identity and $D = d$ at x.

Theorem 3.2. *Assume that $\sigma < \rho < \sigma_0$. Then*

$$f_2^2(\xi,\sigma) \leq e^{w(\rho^2-\sigma^2)}(f_2^2(\xi,\rho) + Q_\infty^2\rho^4)$$

where $w = 4w(n)K_0 + 1$ for $w(n) \sim n!$ a combinatorial constant.

Proof. Let $\varphi(x) = \frac{1}{2}\delta(\xi,x) = \frac{1}{2}|x|^2$ in Gaussian normal coordinates centered at $\xi \in M$. Integrate the inequality of Lemma 3.1 over a geodesic ball of radius σ. From the coordinates given in (9) on the size of the ball

$$W = \sum_j -dx^j \otimes dx^j + \tfrac{1}{4}\frac{\partial}{\partial x^i}(\sqrt{g}\, g^{ij})x^j g_{k\ell}\, dx^k \otimes dx^\ell$$

$$\leq (-\frac{n}{4} + 1 + w(n)K_0|x|^2)g_{k\ell}\, dx^k \otimes dx^\ell\,.$$

Here $w(n)$ depends on the number of terms in $\frac{\partial}{\partial x^i}(\sqrt{g}\, g^{ij})$. We have three further estimates we will insert in the inequality as follows:

$$(W,F^2)_g \leq (-\frac{n-4}{4} + w(n)K_0\sigma^2)|F|^2\,,$$

$$(d\varphi \otimes Q, F) \leq |F|\,|Q|\,|x| \leq \frac{|x|^2}{4}|F|^2 + Q_\infty^2$$

$$\int_{\delta(x,\xi)=\sigma} (x \otimes d\varphi,\, F^2)\, dS \geq 0\,.$$

From the integration over the ball $B_\sigma(\xi)$ we obtain directly

$$\int_{B_\sigma(\xi)} (W,F^*) * 1 = \int_{B_\sigma(\xi)} (d\varphi \otimes Q, F) * 1 \quad + \quad \sigma^{-1} \int_{\delta(x,\xi)=\sigma} (x \otimes d\varphi, F^2)\, dS$$

$$-\frac{\sigma}{4}\int_{\delta(x,\xi)=\sigma} |F|^2\, dS\,.$$

This is converted to the following inequality by substituting the three above inequalities and rearranging a little

$$(\frac{n-4}{4} - \frac{\sigma^2}{4} - w(n)K_0\sigma^2)\int_{B_\sigma(x)} |F|^2 * 1 \leq \frac{\sigma}{4}\int_{\delta(x,\xi)=\sigma} |F|^2\, dS + \sigma^n Q_\infty^2 v_n\,.$$

The constant v_n is essentially the normalized unit ball volume

$$v_n = \max_{\xi, \sigma \le \sigma_0} \sigma^{-n} \int_{B_\sigma(\xi)} *1 .$$

A little manipulation of this inequality converts it into a differential inequality of the form

$$0 \le \frac{d}{d\sigma}\left(\sigma^{-n+4} \exp w\sigma^2 \left(\int_{B_\sigma(\xi)} |F|^2 * 1 \right) + v_n \sigma^4 Q_\infty^2 \right) .$$

Here $w = 1 + 4w(n)K_0$ as we said before. Next integration from σ to ρ yields an inequality which immediately implies the statement in the lemma.

The formulas in the Hermitian-Einstein case for monotonicity are even easier to describe. We work in holomorphic coordinate charts as described in chapter 2. It is a standard fact that globally the energy $\int_M |F|^2 \frac{\omega^m}{m!}$ of a holomorphic connection is given in terms of $\int_M |H|^2 \frac{\omega^m}{m!}$ and a characteristic class term $\int_M \operatorname{tr} F \wedge F \wedge \omega^{m-2}$ because the two-form F is of type (1,1) [U-Y]. This will automatically convert into a monotonicity formula of the type we want, where in the local framework the Chern-Weil term will be replaced by an integral over the boundary.

Lemma 3.3. *Let F be a tensor of the type of a curvature tensor for a unitary connection on a complex Kähler manifold M. Let $H = (\omega, F)$ be the Hermitian-Einstein part and $\tilde{F} \in \Omega^{0,2}(E) + \Omega^{2,0}(E)$ be the part of F which is not of type $(1,1)$. Then if we compute in the metric from the Kähler form*

$$(|F|^2 - |H|^2 - 2|\tilde{F}|^2) * 1 = -\frac{(2i)^{-m}}{(m-2)!} \operatorname{tr} F \wedge F \wedge \omega^{m-2} .$$

Proof. Choose holomorphic coordinates at an arbitrary point $\xi \in M$ in which $\omega = \sum_\alpha dz^\alpha \wedge dz^{\bar\alpha} = \sum_\alpha (2i)\, dy^\alpha \wedge dx^\alpha$. It is not very difficult to see that

$$\operatorname{tr} F \wedge F \wedge \omega^{m-2} = \operatorname{tr} \sum_{\alpha \ne \beta} (F_{\alpha\bar\alpha} F_{\beta\bar\beta} - F_{\alpha\bar\beta} F_{\beta\bar\alpha} + F_{\alpha\beta} F_{\bar\beta\bar\alpha})\, dz^\alpha \wedge dz^{\bar\alpha} \wedge dz^\beta \wedge dz^{\bar\beta} \wedge \omega^{m-2} .$$

The term $dz^\alpha \wedge dz^{\bar\alpha} \wedge dz^\beta \wedge dz^{\bar\beta} \wedge \omega^{m-2}$ represents a form of the top (real) degree $2m = n$ and it is a calculation to show it is $-(m-2)!(2i)^m$ times the standard volume form. Due to the fact that F has the reality properties of the curvature of a unitary connection

$$F^*_{\alpha\bar{\beta}} = F_{\beta\bar{\alpha}} .$$

This gives the identity

$$\operatorname{tr} F \wedge F \wedge \omega^{m-2} = (2i)^m (m-2)! \big(|H|^2 + |\tilde{F}|^2 - |F'|^2\big) * 1$$

where $F' = \sum F_{\bar{\alpha}\beta} dz^{\bar{\alpha}} \wedge dz^{\beta}$ is the part of F of type (1,1) and $\tilde{F} = \sum F_{\alpha,\beta} dz^{\alpha} \wedge dz^{\beta} + F_{\bar{\alpha}\bar{\beta}} dz^{\bar{\alpha}} \wedge dz^{\bar{\beta}}$ is the part of type (0,2) and (2,0). As before $H = \sum_{\alpha} F_{\alpha\bar{\alpha}}$. The lemma follows.

For $\xi \in M$, we work in the coordinate system (21).

Lemma 3.4. *If F is the curvature of a unitary, holomorphic connection, then for $\sigma \leq \sigma_0$*

$$\int_{B_\sigma(\xi)} |F_A|^2 * 1 \leq \int_{B_\sigma(\xi)} |H|^2 * 1 + \frac{\sigma(1+K\sigma^2)}{2(m-2)} \int_{\delta(x,\xi)=\sigma} |F_A|^2 \, dS .$$

Here $B_\sigma(\xi)$ represents the ball of radius σ in our holomorphic coordinate chart based at ξ.

Proof. We use the previous lemma to obtain

$$\int_{B_\sigma(\xi)} |F_A|^2 * 1 = \int_{B_\sigma(\xi)} |H|^2 * 1 + T(m) \int_{B_\sigma(\xi)} \operatorname{tr} F \wedge F \wedge \omega^{m-2} .$$

Here the constant $T(m)$ could be important; however our method of evaluating the Chern-Weil term makes its exact value unimportant. We elect to compute the Chern-Weil term by recalling that $\operatorname{tr} F \wedge F$ is a closed four-form and replace it by $\operatorname{tr}(\tau^* F \wedge \tau^* F)$ where $\tau : B_\sigma(\xi) - \{\xi\} \to \{x : \delta(x,\xi) = \sigma\}$. In our coordinates, $\tau(x) = \frac{\sigma x}{|x|}$. The forms $\operatorname{tr} F \wedge F$ and $\operatorname{tr}(\tau^* F \wedge \tau^* F) = \tau^* \operatorname{tr}(F \wedge F)$ are both closed and have the same tangential components. We apply Lemma 3.3 again to get

$$\int_{B_\sigma(\xi)} |F_A|^2 * 1 = \int_{B_\sigma(\xi)} |H|^2 * 1 + T(m) \operatorname{tr} \tau^* F \wedge \tau^* F \wedge \omega^{m-2}$$

$$\leq \int_{B_\sigma(\xi)} \big(|H|^2 + |\tau^* F_A|^2\big) * 1 .$$

Now, in the case of a flat metric $\omega = \sum_\alpha dz^\alpha \wedge dz^{\bar\alpha}$, the integral over the ball can be precisely evaluated as

$$\int_{B_\sigma(\xi)} |\tau^* F_A|^2 * 1 = \frac{\sigma}{2(m-2)} \int_{\delta(B_\sigma(\xi))} |\tau^* F_A|^2 \, dS ,$$

$$\leq \frac{\sigma}{2(m-2)} \int_{\delta(x,\xi)=\sigma} |F_A|^2 \, dS .$$

The inequality comes from the fact that $\tau^* F_A$ includes only curvatures tangent to $\delta(B_\sigma(\xi))$ and the full curvature has radial components in addition. The $(1 + K\sigma^2)$ must be added to account for deviations of the Kähler metric from the flat case.

Theorem 3.5. *Let F be the curvature of a unitary, holomorphic connection and assume $|H| \leq H_0$. Then if $\sigma < \rho < \sigma_0$*

$$f_2^2(\xi,\sigma) \leq (1 + K\rho^2)^{m-2} f_2^2(\xi,\rho) + \rho^4 v_{2m} H_0^2 (1 + K\sigma_0^2)^{m-3} .$$

Proof. Note $2m = n$, so

$$f_2^2(\xi,\sigma) = \sigma^{-2(m-2)} \int_{B_\sigma(\xi)} |F_A|^2 * 1 .$$

We can rewrite Lemma 3.4 as

$$\sigma^{2(m-2)} f_2^2(\xi,\sigma) \leq v_{2m} \sigma^{2m} H_0^2 + \sigma(1 + K\sigma^2) \frac{d}{d\sigma} (\sigma^{2(m-2)} f_2^2(\xi,\sigma)) .$$

This can be manipulated to give

$$0 \leq \frac{d}{d\sigma} ((1 + K\sigma^2)^{m-2} f_2^2(\xi,\sigma)) + \sigma^3 v_{2m} H_0^2 (1 + K\sigma^2)^{m-3} .$$

We integrate from σ to ρ and make some obvious replacements to get the final result.

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