AUTHOR’S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Rui Philip Xiao
1 Introduction

The K-theory of C*-algebras is the study of a collection of abelian groups
\( K_n(A) \) that are invariants of a C*-algebra \( A \) for \( n \in \mathbb{N} \). In this paper we will
focus on the group \( K_0(A) \). The map \( K_0 \) taking a C*-algebra to an abelian
group can be viewed as a covariant functor from the category of C*-algebras
to the category of abelian groups with some additional properties. We will
follow [4] for this part of the theory.

The K-theory is useful in distinguishing C*-algebras. The class of AF-
algebras is completely classified by their \( K_0 \) groups. In general, the \( K_0 \) group
is not a complete invariant for all C*-algebras, but it is an important part
of the classification program of C*-algebras.

Topological K-theory is the “original version” of K-theory, introduced by
Sir Michael Atiyah. We will follow his classical text [1]. Topological K-theory
is the study of a collection of abelian groups \( K^n(X) \) that are invariants of a
locally compact Hausdorff space \( X \). Unlike the case of C*-algebras, the map
\( K^0 \) is a contravariant functor from the category of locally compact Hausdorff
spaces to the category of abelian groups.

It is well-known that there is a contravariant functor mapping the cate-
gory of unital C*-algebras bijectively onto the category of compact Haus-
dorff spaces that reverses the direction of morphisms. We will see that
\( K^0(X) \cong K_0(C(X)) \) for every compact Hausdorff space \( X \). Furthermore, the
functors \( K_0 \) and \( K^0 \) preserve morphisms by reversing their directions. This
result can be extended to non-unital C*-algebras and locally compact Haus-
dorff spaces, where \( K^0(X) \cong K_0(C_0(X)) \) for every locally compact Hausdorff
space \( X \). This correspondence is explained in [6].

The reader is assumed to be familiar with the basics of C*-algebras and
topological bundles. If one needs a review on these subjects, we recommend

2 K-theory of C*-algebras

Definition 2.1. Let \( A \) be a C*-algebra. For \( n, m \in \mathbb{N} \), let \( M_{m,n}(A) \) be the
set of all \( m \times n \) matrices with entries in \( A \). If \( m = n \), write \( M_{n,n}(A) = M_n(A) \),
then \( M_n(A) \) is a C*-algebra with the involution \( (a^*)_ij = (a_{ji})^* \).

Definition 2.2. Let \( A \) be a C*-algebra. For \( n \in \mathbb{N} \) we define \( P_n(A) \) to be
the set of all projections in \( M_n(A) \). For \( n \leq m \), there is a natural embedding
of \( P_n(A) \) into \( P_m(A) \) given by

\[ p \mapsto \text{Diag}(p, 0_{m-n}) = p \oplus 0_{m-n}. \]

Define \( P_\infty(A) = \lim_{n \to \infty} P_n(A) \) as the direct limit of this inclusion. We can also think of it as \( P_\infty(A) = \bigcup_{n=1}^\infty P_n(A) \).

**Note 2.3.** It might be more notationally clear to write \( p \) as an element in \( P_n(A) \) for \( n \in \mathbb{N} \), and let \([p]\) denote its equivalence class in the direct limit \( P_\infty(A) \). But there are two more equivalence relations to be quotiented by later, and to save ourselves from the nested square brackets, \( p \) will denote a finite matrix as well as its equivalence class in \( P_\infty(A) \), or, an \( \aleph_0 \times \aleph_0 \) matrix with finitely many non-zero entries.

**Definition 2.4.** Let \( \sim_0 \) be the relation on \( P_\infty(A) \) given by the following: for \( p \in P_n(A) \) and \( q \in P_m(A) \), we say \( p \sim_0 q \) if there exists \( v \in M_{m,n}(A) \) such that \( v^*v = p \) and \( vv^* = q \). The relation \( \sim_0 \) is called the Murray - von Neumann equivalence.

**Remark 2.5.** A matrix \( v \in M_{m,n}(A) \) for some \( m, n \in \mathbb{N} \) such that \( v^*v \) and \( vv^* \) are both projections is called a partial isometry. In the special case that \( A = B(H) \) for some Hilbert space \( H \), then \( v \) is a partial isometry if and only if it maps \((\ker v)^\perp \) isometrically onto \( \text{im} v \). If \( T \) is a partial isometry in \( B(H) \), then \( TT^* \) is the projection onto \( \text{im} T \) and \( T^*T \) is the projection onto \((\ker T)^\perp \).

**Example 2.6.** Let \( H \) be an infinite dimensional Hilbert space. Since \( H \cong H \oplus H \), there exists some \( T \in B(H \oplus H) \) such that \( T|_{H \oplus 0} \) is an isometry from \( H \oplus 0 \) onto \( H \oplus H \), and \( T|_{0 \oplus H} = 0 \). Then \( TT^* = I_{H \oplus H} \) and \( T^*T = P_{H \oplus 0} \). Note that \( T \) can be considered as an element in \( B(H \oplus H) \) as well as an element in \( M_2(B(H)) \). In the latter case

\[
\begin{bmatrix}
T_1 & 0 \\
T_2 & 0
\end{bmatrix}
\]

for some \( T_1, T_2 \in B(H) \). If we let \( S = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \), then \( SS^* = I_1 \in M_1(B(H)) \) and \( S^*S = I_2 \in M_2(B(H)) \). So \( I_2 \sim_0 I_1 \).

**Lemma 2.7.** Let \( A \) be a C*-algebra, let \( p \in P_n(A) \) and \( q \in P_m(A) \) for some \( n, m \in \mathbb{N} \), and suppose there exists \( v \in M_{m,n}(A) \) for which \( v^*v = p \) and \( vv^* = q \). Then \( v = qv = vp = qvp \).
Proof. Let \( w = (1 - q)v \), then
\[
  w^*w = v^*(1 - q)(1 - q)v = v^*(1 - q)v = v^*v - vv^*v = p - pp = 0.
\]
However \( \|w\|^2 = \|w^*w\| = 0 \), which implies that \( w = 0 \). So \( 0 = w = v - qv \).
This implies that \( v = qv \). The case \( v = pv \) is proved similarly. Lastly,
\[
  qvp = (qv)p = vp = v.
\]

**Proposition 2.8.** The relation \( \sim_0 \) is an equivalence relation on \( P_\infty(A) \).

**Proof.** It is not yet clear that \( \sim_0 \) is well-defined on \( P_\infty(A) \), since \( P_\infty(A) \) is a
direct limit, where \( p \in P_n(A) \) can also be represented by \( p \oplus 0_k \) in \( P_\infty(A) \), for
any \( k \geq 0 \). We will show that \( \sim_0 \) is an equivalence relation on \( \bigsqcup_{r=1}^\infty P_r(A) \),
and also satisfies \( p \sim_0 p \oplus 0_k \) for \( p \in P_n(A), n \geq 1 \) and \( k \geq 0 \). Then for any
\( p \in P_n(A), q \in P_m(A) \) and \( k, k' \geq 0 \), have \( p \sim_0 q \) if and only if
\[
  p \oplus 0_k \sim_0 p \sim_0 q \sim_0 q \oplus 0_k'.
\]
So \( \sim_0 \) descends to an equivalence relation on \( P_\infty(A) \). To this end, let \( p \in P_n(A), q \in P_m(A) \) and \( r \in P_l(A) \) for some \( l, m, n \geq 1 \).

To show \( p \sim_0 p \oplus 0_k \), let \( v = \begin{bmatrix} p & 0_{n \times k} \end{bmatrix} \), then \( v^*v = p \) and \( vv^* = p \oplus 0_k \).
The special case with \( k = 0 \) verifies reflexivity.

Suppose there exists \( v \in M_{m,n}(A) \) such that \( v^*v = p \) and \( vv^* = q \). Let
\( w = v^* \in M_{n,m}(A) \). We have
\[
  w^*w = q \text{ and } ww^* = p.
\]
So \( \sim_0 \) is symmetric.

Suppose \( p \sim_0 q \) and \( q \sim_0 r \). Then there exists some \( v \in M_{m,n}(A) \) and \( u \in M_{l,m}(A) \) for which
\[
  v^*v = p, \quad vv^* = q, \quad u^*u = q \quad \text{and} \quad uu^* = r
\]
hold. Let \( z = uv \). Using Lemma 2.7, the following computations hold.
\[
  z^*z = v^*u^*uv = v^*qv = v^*v = p,
\]
\[
  zz^* = uvv^*u^* = uqu^* = r.
\]
Thus \( p \sim_0 r \), which proves transitivity. \( \blacksquare \)
Definition 2.9. Let $A$ be a C*-algebra and $p, q$ projections in $\mathcal{P}_\infty(A)$. We say that $p$ and $q$ are mutually orthogonal if $pq = 0$, written $p \perp q$.

Remark 2.10. If $p \perp q$ then
\[ qp = q^*p^* = (pq)^* = 0^* = 0, \]
so $q \perp p$. And also,
\[ (p + q)^* = p^* + q^* = p + q \]
\[ (p + q)(p + q) = pp + pq + qp + qq = pp + qq = p + q. \]
So $p + q$ is also a projection in $A$.

In the special case that $A = B(H)$ for some Hilbert space $H$ and $P, Q \in B(H)$ are projections, we have $P \perp Q$ if and only if ran $P \perp$ ran $Q$.

Proposition 2.11. Let $p, p', q, q' \in \mathcal{P}_\infty(A)$. Then

1. $p \oplus q \sim_0 q \oplus p$.
2. $p \sim_0 p'$ and $q \sim_0 q'$ implies $p \oplus q \sim_0 p' \oplus q'$.
3. $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.
4. Suppose $p$ and $q$ are represented by matrices of the same size, and $p \perp q$, then $p + q \sim_0 p \oplus q$.

Proof. 1. Suppose $p$ is $n \times n$ and $q$ is $m \times m$. Let $v = \begin{bmatrix} 0_{n \times m} & p \\ q & 0_{m \times n} \end{bmatrix}$. Then
\[ v^*v = \begin{bmatrix} 0_{n \times n} & q^* \\ p^* & 0_{m \times m} \end{bmatrix} \begin{bmatrix} 0_{n \times m} & p \\ q & 0_{m \times n} \end{bmatrix} = \begin{bmatrix} q^*q & 0_{n \times n} \\ pp^* & p^*p \end{bmatrix} = q \oplus p; \]
\[ vv^* = \begin{bmatrix} 0_{n \times m} & p \\ q & 0_{m \times n} \end{bmatrix} \begin{bmatrix} 0_{m \times n} & q^* \\ p^* & 0_{n \times m} \end{bmatrix} = \begin{bmatrix} pp^* & 0_{n \times m} \\ qq^* & 0_{m \times n} \end{bmatrix} = q \oplus p. \]
So $q \oplus p \sim_0 p \oplus q$.

2. Suppose $v^*v = p$, $vv^* = p'$, $w^*w = q$ and $ww^* = q'$, then
\[ (v \oplus w)^*(v \oplus w) = p \oplus q \]
and
\[ (v \oplus w)(v \oplus w)^* = p' \oplus q'. \]
So \( p \oplus q \sim_0 p' \oplus q' \).

3. This is by definition.

4. Suppose \( p \) and \( q \) are of the same size and \( pq = 0 \). Let \( v = [p \quad q] \), then

\[
vv^* = \begin{bmatrix} p & q \\ q & p \end{bmatrix} = pp + qq = p + q,
\]

\[
v^*v = \begin{bmatrix} p \\ q \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix} = \begin{bmatrix} pp & pq \\ qp & qq \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = p \oplus q.
\]

So \( p + q \sim_0 p \oplus q \).

**Definition 2.12.** Let \( A \) be a C*-algebra. Define \( \mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0 \). The equivalence class of \( p \) in \( \mathcal{D}(A) \) is written \([p]_\mathcal{D}\). Equip \( \mathcal{D}(A) \) with an operation \(+\) by \([p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}\).

**Proposition 2.13.** \((\mathcal{D}(A), +)\) is an abelian monoid.

**Proof.** This is mostly a consequence of Proposition 2.11. Point 2 implies that the operation \(+\) is well-defined after quotienting by \( \sim_0 \). Point 3 implies that \(+\) is associative. Point 1 implies that it is commutative. So \((\mathcal{D}(A), +)\) is an abelian semigroup. Now we claim that \([0_1]_\mathcal{D}\) is the identity element (note that \(0_n \sim_0 0_m\) for all \(n, m \in \mathbb{N}\) by Proposition 2.8). To this end, take any \( p \in \mathcal{P}_\infty(A) \). By point 1 of Proposition 2.11 and Proposition 2.8,

\[
0_1 \oplus p \sim_0 p \oplus 0_1 \sim_0 p,
\]

so

\[
[0_1]_\mathcal{D} + [p]_\mathcal{D} = [p]_\mathcal{D} + [0_1]_\mathcal{D} = [p]_\mathcal{D}.
\]

From the abelian monoid \( \mathcal{D}(A) \) we will construct an abelian group, by a construction called the **Grothendieck completion**.

**Definition 2.14.** Let \((S, +)\) be an abelian semigroup, then \( S \times S \) is also naturally a semigroup. Let \( \sim \) be a relation on \( S \times S \) given by \((a_1, b_1) \sim (a_2, b_2)\) if there exists \( x \in S \) so that

\[
a_1 + b_2 + x = a_2 + b_1 + x.
\]

Define \( G(S) = (S \times S)/\sim \), and equip it with the operation \(+\) by

\[
[(a, b)] + [(c, d)] = [(a + c, b + d)].
\]
Proposition 2.15. The above construction is well-defined, and $G(S)$ is an abelian group. Furthermore, if $S$ is an abelian monoid with identity element 0, then $\varphi : S \to G(S)$ by $\varphi(s) = [(s, 0)]$ is a monoid homomorphism.

Proof. It is easy to see that $\sim$ is an equivalence relation on $S \times S$. To see that $+$ is well-defined on $G(S)$, let $a_i, b_i, c_i, d_i \in S$ for $i = 1, 2$, and suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$. Then there exists $x, y \in S$ such that

$$a_1 + b_2 + x = a_2 + b_1 + x \quad \text{and} \quad c_1 + d_2 + y = c_2 + d_1 + y.$$

Then

$$(a_1 + c_1) + (b_2 + d_2) + (x + y) = (a_2 + c_2) + (b_2 + d_2) + (x + y),$$

so $[(a_1 + c_1, b_1 + d_1)] = [(a_2 + c_2, b_2 + d_2)]$.

Since $+$ is associative and commutative on $S$, the addition induced on $G(S)$ is associative and commutative as well. For $a, b, c, d \in S$, it is clear that $[(a, a)] = [(b, b)]$. Furthermore

$$[(c, d)] + [(a, a)] = [(c + a, d + a)] = [(c, d)].$$

So $(a, a)$ is the identity element of $G(S)$. Also,

$$[(a, b)] + [(b, a)] = [(a + b, a + b)],$$

so $[(b, a)]$ is the inverse of $[(a, b)]$. Hence $G(S)$ is indeed an abelian group.

Now suppose that $S$ is an abelian monoid with 0, and $\varphi : S \to G(S)$ by $\varphi(s) = [(s, 0)]$. Then it is clear that $\varphi(a + b) = \varphi(a) + \varphi(b)$ and that $\varphi(0)$ is the identity element of $G(S)$. □

It is convenient to think of $[(a, b)] \in G(S)$ as “$a - b$”.

Example 2.16. 1. $S = \mathbb{N}$. Then $G(\mathbb{N}) = \mathbb{Z}$. This is the standard construction of $\mathbb{Z}$.

2. $S = \mathbb{N} \cup \{\infty\}$. For any $a, b, c, d \in \mathbb{N} \cup \{\infty\}$,

$$a + c + \infty = \infty = b + d + \infty,$$

so $[(a, b)] = [(c, d)]$. Hence $G(S) \cong \{0\}$. This example demonstrates why we required the $x$ in defining $\sim$ in Definition 2.14, where $(a_1, b_1) \sim (a_2, b_2)$
if and only if there exists $x$ for which $a_1 + b_2 + x = a_2 + b_1 + x$.

Suppose for instance we define another relation $\sim_{\text{bad}}$ on $S$ by $(a_1, b_1) \sim_{\text{bad}} (a_2, b_2)$ if $a_1 + b_2 = a_2 + b_1$. For any $a, b \in S$, we have

$$\infty + a = \infty = b + \infty,$$

so $(\infty, \infty) \sim_{\text{bad}} (a, b)$. In particular, $(1, 1) \sim_{\text{bad}} (\infty, \infty) \sim_{\text{bad}} (1, 2)$, but clearly $(1, 1) \not\sim_{\text{bad}} (1, 2)$, which shows that $\sim_{\text{bad}}$ is not an equivalence relation!

This is the same problem that one runs into when asking “Surely $\infty + \infty = \infty$, but what is $\infty - \infty$?”

Now we are ready to give the definition of the $K_0$ group of a unital C*-algebra.

**Definition 2.17.** Let $A$ be a unital C*-algebra. Define $K_0(A) = G(D(A))$.

Define the map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$ by $[p]_0 = \varphi([p]_D)$ where $\varphi : D(A) \to G(D(A))$ is the monoid homomorphism defined in Proposition 2.15.

**Example 2.18.** 1. Let $A = \mathbb{C}$. All projections in $\mathcal{P}_\infty(\mathbb{C})$ are projection matrices. Take $p, q \in \mathcal{P}_\infty(\mathbb{C})$. We may assume that $p$ and $q$ are both $n \times n$. Suppose $p$ and $q$ have the same rank $k \leq n$, and let $\{z_1, \ldots, z_k\}$ be an orthonormal basis of $\text{ran} p$ and extend it to an orthonormal basis $\{z_1, \ldots, z_n\}$ of $\mathbb{C}^n$; let $\{w_1, \ldots, w_k\}$ be an orthonormal basis of $\text{ran} q$. Let $v \in M_n(\mathbb{C})$ be the matrix that takes $z_j$ to $w_j$ for $j = 1, \ldots, k$, and takes $z_j$ to 0 for all $j = k + 1, \ldots, n$. Then

$$v^*vz_j = \begin{cases} v^*w_j = z_j & : j = 1, \ldots, k \\ v^*0 = 0 & : j = k + 1, \ldots, n. \end{cases}$$

So $v^*v$ is the projection onto $\text{ran} p$, hence $v^*v = p$. Similarly, $vv^* = q$, so $p \sim_0 q$.

Conversely suppose $p \sim_0 q$. Then there exits a matrix $v$ for which $v^*v = p$ and $vv^* = q$. Since row rank and column rank coincide, we have

$$\text{rank } p = \text{rank } v^*v = \text{rank } vv^* = \text{rank } q.$$
\[ K_0(\mathbb{C}) \cong G(\mathbb{N}) = \mathbb{Z}. \]

2. Let \( A = M_m(\mathbb{C}) \) for some \( m \in \mathbb{N} \). Then for \( n \in \mathbb{N} \), the C*-algebra \( M_n(A) \) is naturally a subalgebra of \( M_{mn}(\mathbb{C}) \), and the rank argument from above works just as well. Hence \( K_0(M_m(\mathbb{C})) \cong \mathbb{Z} \).

3. Let \( A = \mathcal{B}(\mathcal{H}) \) for \( \mathcal{H} \) an infinite dimensional Hilbert space. The same rank argument works since every two Hilbert spaces of the same dimension are isometric. So projections in \( \mathcal{P}_\infty(A) \) are once again determined up to Murray - von Neumann equivalence by their dimensions, and \( \mathcal{D}(A) \cong \{ \dim p : p \in \mathcal{P}_\infty(A) \} \). Since \( \mathcal{H} \) is infinite dimensional, \( \mathcal{D}(A) \) has a largest element \( \alpha_0 = \dim \mathcal{H} \) since \( \dim(\mathcal{H}^n) = \dim \mathcal{H} \) for all finite \( n \), and \( \alpha_0 + \alpha = \alpha_0 \) for all \( \alpha \in \mathcal{D}(A) \). So by the same argument in part 2 of Example 2.16, have \( K_0(\mathcal{B}(\mathcal{H})) = G(\mathcal{D}(\mathcal{B}(\mathcal{H}))) = 0 \).

To summarize,

\[
K_0(\mathcal{B}(\mathcal{H})) \cong \begin{cases} \mathbb{Z} & : \dim \mathcal{H} < \aleph_0 \\ 0 & : \dim \mathcal{H} \geq \aleph_0 \end{cases}
\]

3 Unitaries and projections

In this section we develop some properties of unitary and projection elements in a C*-algebra. These will be necessary for exploring meaningful properties of the \( K_0 \)-group of C*-algebras.

From here on \( \tilde{A} \) denotes the unitization of the C*-algebra \( A \). For more information on unitization, see [2].

**Definition 3.1.** Let \( X \) be a topological space and \( x, y \in X \). Say \( x \) and \( y \) are **homotopy equivalent** in \( X \), written \( x \sim_h y \), if there exists a continuous path \( \alpha : [0, 1] \to X \) such that \( \alpha(0) = x \) and \( \alpha(1) = y \).

**Definition 3.2.** Let \( A \) be a C*-algebra, and \( a, b \in A \). We say \( a \) is **unitarily equivalent** to \( b \), written \( a \sim_u b \), if there exists a unitary \( u \in \tilde{A} \) such that \( uau^* = b \). It is clear that these are equivalence relations.
Definition 3.3. Let $A$ be a unital C*-algebra, define $\mathcal{U}(A)$ to be the group of unitary elements in $A$, and define $\mathcal{U}_0(A)$ to be all $u \in \mathcal{U}(A)$ such that $u \sim_h 1$. That is, $\mathcal{U}_0(A)$ is the path-connected component of 1 in $\mathcal{U}(A)$.

Definition 3.4. Let $A$ be a unital C*-algebra and let $a \in A$. The spectrum $\sigma(a)$ of $a$ is defined to be

$$\sigma(a) := \{ \lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } A \}.$$ 

The general theory of spectrum and of continuous functional calculus can be found in [2].

Lemma 3.5. Let $A$ be a unital C*-algebra and $u \in \mathcal{U}(A)$. If $\sigma(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(A)$.

Proof. Suppose $\sigma(u) \neq \mathbb{T}$. Let $w \in \mathbb{T} \setminus \sigma(u)$ and let $\log_w : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ be the branch of logarithm that avoids the ray containing $w$. Then $\exp(\log_w(z)) = z$ for all $z \in \mathbb{T} \setminus \{w\} \supseteq \sigma(u)$, so $\exp(\log_w(u)) = u$. Let $h = \log_w(u)$, then

$$\sigma(h) \subseteq \log_w(\mathbb{T} \setminus w) \subseteq i\mathbb{R}.$$ 

For $t \in [0, 1]$, let $h_t = th$. Clearly $\sigma(th) \subseteq i\mathbb{R}$ for all $t \in [0, 1]$, so $\sigma(\exp(th)) \subseteq \mathbb{T}$ for all $t \in [0, 1]$, which implies that $\exp(th)$ is unitary for any $t \in [0, 1]$. Furthermore the map $\beta : [0, 1] \to \mathcal{U}(A)$ mapping $\beta(t) = \exp(th)$ is a continuous path of unitaries from $1_A \in A$ to $u \in A$. Hence $u \in \mathcal{U}_0(A)$. 

Lemma 3.6 (Whitehead). Let $A$ be a unital C*-algebra and let $u, v \in \mathcal{U}(A)$. Then

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \sim_h \begin{bmatrix} uv & 0 \\ 0 & v \end{bmatrix} \sim_h \begin{bmatrix} vu & 0 \\ 0 & v \end{bmatrix} \sim_h \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} \text{ in } \mathcal{U}(M_2(A)).$$

Proof. Since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has spectrum $\{\pm 1\}$, by Lemma 3.5 have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim_h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
Let $\alpha : [0, 1] \to \mathcal{U}_0(M_2(A))$ be a path from $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Define $\beta : [0, 1] \to M_2(A)$ by

$$\beta(t) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \alpha(t) \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \alpha(t).$$

Since for all $t \in [0, 1]$, $\beta(t)$ is the product of four unitaries, so $\beta$ is in fact a path in $\mathcal{U}(M_2(A))$. Further,

$$\beta(0) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix},$$

and

$$\beta(1) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} uv & 0 \\ 0 & 1 \end{bmatrix}.$$ 

So

$$\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \sim_h \begin{bmatrix} uv & 0 \\ 0 & 1 \end{bmatrix}.$$

By symmetry and transitivity, it is only left to prove that

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \sim_h \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}.$$

This can be accomplished by defining the path

$$\gamma(t) = \alpha(t) \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \alpha(t).$$

**Corollary 3.7.** Let $A$ be a unital $C^*$-algebra, $u \in \mathcal{U}(A)$, then $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \in U_0(M_2(A))$.

**Proof.** By Lemma 3.6,

$$\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \sim_h \begin{bmatrix} uu^* & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
Lemma 3.8. Let \( A \) be a unital C*-algebra and \( u \in \mathcal{U}(A) \). If \( \| u - 1 \| < 2 \) then \( u = \exp(ih) \) for some self-adjoint element \( h \in A \).

Proof. If \( \| u - 1 \| < 2 \) then \( \sigma(u - 1) \subseteq B_2(0) \), in particular \( -2 \notin \sigma(u - 1) \), so \( -1 \notin \sigma(u) \). Since \( \sigma(u) \neq \mathbb{T} \), by the proof of Lemma 3.5, \( u = \exp(s) \) for some \( s \in A \) with \( \sigma(s) \in i\mathbb{R} \). Let \( h = -is \), then \( h \) is self-adjoint and \( \exp(ih) = \exp(s) = u \).

Proposition 3.9. Let \( A \) be a unital C*-algebra. Then

\[
\mathcal{U}_0(A) = \{ \exp(ih_1) \ldots \exp(ih_l) : l \in \mathbb{N}, h_j \in A \text{ self-adjoint} \}.
\]

Proof. Let \( u \in \mathcal{U}_0(A) \). A continuous path from \( u \) to 1 can be partitioned into segments

\[
u = u_0 \sim_h u_1 \sim_h \ldots \sim_h u_k = 1
\]

where \( \| u_{j-1} - u_j \| < 2 \) for \( j = 1, \ldots, k \). Now apply induction on \( k \). For \( k = 1, \| u - 1 \| < 2 \), and the result follows Lemma 3.8. Suppose the result is true for \( k = n - 1 \), and the inductive step for \( n \) has been completed. Then \( u_1 = \exp(ih_1) \ldots \exp(ih_l) \) for some \( l \in \mathbb{N} \) and \( h_j \) self-adjoint. Because \( \| u - u_1 \| < 2 \), so

\[
\| uu_1^* - 1 \| = \|(u - u_1)u_1^*\| = \|u - u_1\| < 2.
\]

By Lemma 3.8, there exists a self-adjoint element \( h_0 \in A \) such that \( uu_1^* = \exp(ih_0) \). Then

\[
u = \exp(ih_0)u_1 = \exp(ih_0)\exp(ih_1) \ldots \exp(ih_l).
\]

This completes the induction.

Conversely if \( h \) is self-adjoint, the proof of Lemma 3.5 implies that \( \exp(ih) \in \mathcal{U}_0(A) \). The product of such unitaries is also homotopic to the identity. Thus all elements in \( \mathcal{U}_0(A) \) are indeed equal to finite products as in the claim. \( \blacksquare \)

Proposition 3.10. Let \( A, B \) be unital C*-algebras, \( \varphi : A \to B \) a surjective *-homomorphism. Then

1. \( \varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B) \)
2. For any $u \in U(B)$, there exists $v \in U_0(M_2(A))$ such that

$$\varphi(v) = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$$

Proof. 1. Since $\varphi$ takes unitaries to unitaries, $\varphi(U_0(A)) \subseteq U_0(B)$. The converse requires some work. Let $u \in U_0(B)$. By Proposition 3.9, there exists hermitian elements $h_1, \ldots, h_l \in B$ such that

$$u = \exp(ih_1) \exp(ih_2) \ldots \exp(ih_l).$$

Let $t_1, \ldots, t_l \in A$ such that $\varphi(t_j) = h_j$ for $j = 1, \ldots, l$, and let $\tilde{t}_j = \frac{1}{2}(t_j + t_j^*)$ for $j = 1, \ldots, l$. Then $\tilde{t}_j$ are self-adjoint, and

$$\varphi(\tilde{t}_j) = \frac{1}{2}(\varphi(t_j) + \varphi(t_j)^*) = \frac{1}{2}(h_j + h_j) = h_j.$$

Let

$$v = \exp(i\tilde{t}_1) \ldots \exp(i\tilde{t}_l).$$

The proof of Lemma 3.5 implies that $v \in U_0(A)$. And happily, $\varphi(v) = u$.

2. Let $u \in U(B)$. By Corollary 3.7 $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \in U_0(M_2(B))$. Then by part 1 there exists some $v \in U_0(M_2(A))$ such that $\varphi(v) = u \oplus u^*$. ■

Definition 3.11. Let $A$ be a unital C*-algebra and $a \in A$. Then $\sigma(a^*a) \subseteq \mathbb{R}_{\geq 0}$, where the square root function is defined. So we may define $|a| = (a^*a)^{1/2}$.

Proposition 3.12. Let $A$ be a unital C*-algebra.

1. If $z \in GL(A)$, then $|z| \in GL(A)$, and $w(z) := z|z|^{-1} \in U(A)$.

2. The map $w : GL(A) \to U(A)$ defined in 1. is continuous. And $w(u) = u$ for all $u \in U(A)$.

3. If $a, b \in GL(A)$ with $a \sim_h b$ in $GL(A)$, then $w(a) \sim_h w(b)$ in $U(A)$. 

12
Proof. 1. Suppose $z$ is invertible. Then $z^*$ is also invertible, so $z^*z \in GL(A)$. It follows that

$$\sigma(|z|) = \sigma((z^*z)^{1/2}) = \{t^{1/2} : t \in \sigma(z^*z)\} \neq 0.$$ 

Thus $|z|$ is invertible.

Furthermore, $w(z)w(z)^* = z|z|^{-1}(z|z|^{-1})^* = z|z|^{-1}|z|^{-1}z^*$

$$= z(z^*z)^{-1}z^* = zz^{-1}(z^*)^{-1}z^* = 1,$$

and similarly $w(z)^*w(z) = 1$. So $w(z) \in U(A)$.

2. The map $a \mapsto a^*a$ is continuous. Also inversion and multiplication are continuous in $GL(A)$. So to prove the claim it is sufficient to prove that $a \mapsto a^{1/2}$ is continuous on $A_{\geq 0}$, where $A_{\geq 0}$ is the set of normal elements in $A$ with spectrum contained in $[0, \infty)$.

Suppose we fix $a \in A_{\geq 0}$ and let $U$ be a bounded open neighbourhood containing $\sigma(a)$. The upper-semicontinuity of spectra [5] implies that there is some $d > 0$ such that if $b \in A$ and $\|b - a\| < d$ then $\sigma(b) \subseteq U$. Thus the problem reduces to proving that the square root map is continuous on $\Omega_r \subseteq A_{\geq 0}$ where

$$\Omega_r = \{a \in A : a^*a = aa^*, \sigma(a) \subseteq [0, r]\}.$$ 

Let $f$ denote the square root function and let $\varepsilon > 0$ be given. By the Stone-Weierstrass theorem, there exists a complex polynomial $g$ such that $\|g - f\|_\infty < \varepsilon/3$ on $[0, r]$. For $c \in \Omega_t$,

$$\|f(c) - g(c)\| = \|(f - g)(c)\|$$

$$= \sup\{|(f - g)(z)| : z \in \sigma(c)\}$$

$$\leq \|f - g\|_\infty < \varepsilon/3.$$ 

Therefore $g$ is continuous on $\Omega_t$ since $a \mapsto a^n$ is continuous. So there exists $\delta > 0$ such that $\|g(a) - g(b)\| < \varepsilon/3$ whenever $a, b \in A$ with $\|a - b\| < \delta$. Thus when $a, b \in \Omega_t$ with $\|a - b\| < \delta$, have $\|f(a) - f(b)\| < \varepsilon$.

3. Let $\alpha : [0, 1] \to GL(A)$ be a continuous path from $a$ to $b$. Then by part 2, $w \circ \alpha : [0, 1] \to U(A)$ is a continuous path from $w(a)$ to $w(b)$.

For an element $z \in A$, the form $z = w(z)|z|$ is called the polar decomposition of $z$.  

13
Definition 3.13. The relations $\sim_u$ and $\sim_h$ induce equivalence relations on $\mathcal{P}_\infty(A)$ as follows: $p \sim_u q$, if by representing $p$ and $q$ both as $n \times n$ matrices for some $n \in \mathbb{N}$, there exists a unitary element $u \in M_n(A)$ such that $u^*pu = q$. We say that $p \sim_h q$ if by representing $p$ and $q$ both as $n \times n$ matrices for some $n \in \mathbb{N}$, there exists a path $\alpha(t)$ in $\mathcal{P}_n(A)$ such that $\alpha(0) = p$ and $\alpha(1) = q$.

Proposition 3.14. Let $A$ be a unital $C^*$-algebra, $a, b \in A$ self-adjoint elements, $z \in GL(A)$ and $z = u|z|$ the polar decomposition of $z$. If $za = bz$ then $ua = bu$.

Proof. Since $a$ and $b$ are self-adjoint, take the adjoint of the equality to have $az^* = z^*b$. Then

$$|z|^2a = z^*za = z^*bz = a^*z = a|z|^2.$$ 

So $a$ commutes with $|z|^2$. Consequently $a$ commutes with $g(|z|^2)$ for all complex polynomials $g$. By Stone-Weierstrass theorem, the element $|z|^{-1} = ((|z|^2)^{1/2})^{-1}$ is the limit of a sequence of polynomials in $|z|^2$. Hence $a$ commutes with $|z|^{-1}$. It follows that

$$uau^* = z|z|^{-1}a u^* = z a |z|^{-1}u^* = b z |z|^{-1}u^* = bu u^* = b.$$ 

Proposition 3.15. Let $n \in \mathbb{N}_{\geq 1}$, and $p, q \in \mathcal{P}_n(A)$. Then

1. $p \sim_h q$ implies $p \sim_u q$.
2. $p \sim_u q$ implies $p \sim_0 q$.
3. $p \sim_0 q$ implies $p \oplus I_n \sim_u q \oplus I_n$.
4. $p \sim_u q$ implies $p \oplus I_n \sim_h q \oplus I_n$.

Proof. 1. Let $\alpha(t)$ be a path in $\mathcal{P}_n(A)$ that connects $p$ to $q$, then we can partition the path into segments of length less than $1/2$. It is now sufficient to prove that if $\|p - q\| < 1/2$ then $p \sim_u q$. Let $z = pq + (I - p)(I - q) \in \hat{A}$, and $pz = pq = zq$. Also

$$\|z - I\| = \|pq + (I - p)(I - q) - I\|$$

$$= \|pq + (I - p)(I - q) - p - (I - p)\|$$

$$= \|p(q - p) + (I - p)((I - q) - (I - p))\|$$

$$= \|p(q - p) + (I - p)(p - q)\|$$

$$\leq \|p\|\|(q - p)\| + \|I - p\|\|p - q\|$$

$$\leq 2\|p - q\| < 1.$$
Hence $z \in GL(A)$. Let $z = u|z|$ be the polar decomposition of $z$. By Proposition 3.14, $pu = uq$.

2. Suppose $p \sim_u q$. Then there exists some unitary $u \in \widehat{M_n(A)}$ such that $u^*pu = q$. Let $v = u^*p$, then $vv^* = u^*ppu = q$ and $v^*v = puu^*p = pp = p$. Also note that $v = u^*p \in M_n(A)$ since $M_n(A)$ is an ideal in $\widehat{M_n(A)}$. Hence $p \sim_0 q$.

3. Suppose there exists $v \in M_n(A)$ such that $vv^* = q$ and $v^*v = p$. Define

$$u = \begin{bmatrix} v & 1 - q \\ 1 - p & v^* \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} q & 1 - q \\ 1 - q & q \end{bmatrix}. $$

Then

$$u^*u = \begin{bmatrix} v & I_n - q \\ I_n - p & v^* \end{bmatrix} \begin{bmatrix} v^* & I_n - p \\ I_n - q & v \end{bmatrix} = \begin{bmatrix} vv^* + (I_n - q) & v - vp + v - qv \\ v^* - pv^* + v^* - v^*q & (I_n - p) + v^*v \end{bmatrix} = \begin{bmatrix} I_n + q - q & v - v + vv^*v - vv^*v \\ v^* - v^* + v^*vv^* - v^*v v^* & I_n - v^*v + v^*v \end{bmatrix} = I_{2n}.$$ 

Lemma 2.7 is used to equate the second line to the third in the above equation. Similar computations show that $uu^* = w^*w = ww^* = I_{2n}$. So $u, w, wu \in U_{2n}(\mathcal{A})$. And

$$wu = \begin{bmatrix} q & I - q \\ I - q & q \end{bmatrix} \begin{bmatrix} v & I - q \\ I - p & v^* \end{bmatrix} = \begin{bmatrix} qv + (I - q)(I - p) & q - qq + v^* - qv^* \\ v - qv + q - qp & (I - q)(I - q) + qv^* \end{bmatrix} = \begin{bmatrix} v + (I - q)(I - p) & (I - q)v^* \\ q(I - p) & (I - q) + qv^* \end{bmatrix}. $$

15
is an element of $\widetilde{M}_{2n}(A)$. Now,

$$wu(p \oplus 0_n)(wu)^*$$

$$= \begin{bmatrix} v+(I-q)(I-p) & 0 \\ q-pq & I-q+v^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v^*+(I-p)(I-q) & q-pq \\ 0 & I-q+v \end{bmatrix}$$

$$= \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v^*+(I-p)(I-q) & q-pq \\ 0 & I-q+v \end{bmatrix}$$

$$= \begin{bmatrix} vv^* + v(I-p)(I-q) & vq - vpq \\ 0 & 0 \end{bmatrix} = q \oplus 0_n$$

noting that

$$v(I-p)(I-q) = (v - vv^*v)(I-q) = 0$$

and

$$vq - vpq = vv^* - (vv^*v)vv^* = vv^* - vv^* = 0$$

by Lemma 2.7.

4. Suppose $p \sim_u q$. Then there exists unitary $u \in \widetilde{M}_n(A)$ such that $upu^* = q$. By Lemma 3.6 there exists a path $t \mapsto w_t$ in $\mathcal{U}(\widetilde{M}_{2n}(A))$ such that

$$w_0 \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \text{ and } w_1 \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}.$$  

Let $p_t = w_t \text{Diag}(p, 0_n) w^*_t$. Then $p_t \in \mathcal{P}_{2n}(A)$ for each $t \in [0, 1]$. Furthermore,

$$p_0 = \text{Diag}(p, 0_n) \text{ and } p_1 = \begin{bmatrix} upu^* & 0 \\ 0 & 0 \end{bmatrix} = \text{Diag}(q, 0_n).$$

Therefore $p \oplus 0_n \sim_h q \oplus 0_n$. 

\section{\textit{K}_0 \text{ as a functor}}

We will see that $K_0$ is a contravariant functor from the category of C*-algebras to the category of abelian groups, and that it enjoys many useful properties. Before starting the functoriality, we will first need a way to induce group homomorphisms from semigroups homomorphisms in the Grothendieck completion.
Proposition 4.1. Let $S$ be an abelian semigroup. For any abelian group $H$ and any semigroup homomorphism $\rho : S \to H$, the map $\rho_G : G(S) \to H$ given by $\rho_G([(s,t)]_G) = \rho(s) - \rho(t)$ for all $(s,t) \in S \times S$ is a well-defined group homomorphism.

Proof. Let $\rho_G$ be as defined above and let $s_1, s_2, t_1, t_2 \in S$. To see that $\rho_G$ is well-defined, suppose that $[(s_1,t_1)]_0 = [(s_2,t_2)]_0$. Then there exists $r \in S$ such that $s_1 + t_2 + r = s_2 + t_1 + r$, which implies that

$$\rho(s_1) + \rho(t_2) + \rho(r) = \rho(s_2) + \rho(t_1) + \rho(r).$$

But $H$ is a group, where all elements are invertible. So

$$\rho_G([(s_1,t_1)]_G) = \rho(s_1) - \rho(t_1) = \rho(s_2) - \rho(t_2) = \rho_G([(s_2,t_2)]_G).$$

Hence $\rho_G$ is well-defined. Now to check that $\rho_G$ is a homomorphism:

$$\rho_G([(s_1,t_1)]_G + [(s_2,t_2)]_G) = \rho_G([(s_1+s_2,t_1+t_2)]_G)$$
$$= \rho(s_1 + s_2) - \rho(t_1 + t_2)$$
$$= (\rho(s_1) - \rho(t_1)) + (\rho(s_2) - \rho(t_2))$$
$$= \rho_G([(s_1,t_1)]_0) + \rho_G([(s_2,t_2)]_0).$$

If $A$ and $B$ are C*-algebras, with $\varphi : A \to B$ a continuous $*$-homomorphism, then $\varphi$ extends naturally to a $*$-homomorphism $M_n(A) \to M_n(B)$ for all $n \in \mathbb{N}$ by applying $\varphi$ entry-wise to matrix entries, i.e. $\varphi(T)_{ij} = \varphi(T_{ij})$. This map clearly respects matrix multiplication and involution. In the same way, $\varphi$ extends entry-wise to $\mathcal{P}_\infty(A)$ and respects direct sum, and is thus a monoid homomorphism $\mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)$. Let $\pi : \mathcal{P}_\infty(B) \to \mathcal{P}_\infty(B)/\sim_0$ be the quotient map. Then $\pi \circ \varphi$ is a monoid homomorphism $\mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)/\sim_0$. If $p, q \in \mathcal{P}_\infty(A)$ with $p \sim_0 q$, there exists some matrix $v$ with entries in $A$ such that $vv^* = p$ and $v^*v = q$. Hence

$$\pi \circ \varphi(p) = \pi(\varphi(vv^*)) = \pi(\varphi(v)\varphi(v^*))$$
$$= \pi(\varphi(v^*)\varphi(v)) = \pi(\varphi(v^*v))$$
$$= \pi \circ \varphi(q).$$

So $\pi \circ \varphi(p)$ factors into a monoid homomorphism $\tilde{\varphi} : \mathcal{P}_\infty(A)/\sim_0 \to \mathcal{P}_\infty(B)/\sim_0$ by $\tilde{\varphi}([p]) = \pi \circ \varphi(p) = [\varphi(p)]$. 

17
Proposition 4.2. Let $A$ and $B$ be $C^*$-algebras and $\varphi : A \to B$ a continuous $*$-homomorphism. Then there exists a group homomorphism $K_0(\varphi) : A \to B$ satisfying $K_0(\varphi)([p]_0) = [\varphi(p)]_0$ for all $p \in \mathcal{P}_\infty(A)$.

Proof. Recall that $K_0(A) = G(\mathcal{P}_\infty(A) / \sim_0)$, where there is a monoid homomorphism $[.]_0 : A \to K_0(A)$. By the previous paragraph, we have a monoid homomorphism
\[
\tilde{\varphi} : \mathcal{P}_\infty(A) / \sim_0 \to \mathcal{P}_\infty(B) / \sim_0.
\]
By Proposition 4.1, let $K_0 = \tilde{\varphi}_G$; and let $\iota_A, \iota_B$ be the “inclusion” from $\mathcal{D}(A) \to K_0(A)$ and $\mathcal{D}(B) \to K_0(B)$ respectively, as in Proposition 2.14. Then
\[
K_0(\varphi)([p]_0) = K_0(\varphi)(\iota_A([p]_\mathcal{D})) = \tilde{\varphi}_G(([p]_\mathcal{D}, [0]_\mathcal{D})_G)
= ([\tilde{\varphi}(p)_\mathcal{D}, \tilde{\varphi}(0)_\mathcal{D})_G = \iota_B \circ \tilde{\varphi}([p]_\mathcal{D})
= \iota_B([\varphi(p)]_\mathcal{D}) = [\varphi(p)]_0.
\]

Proposition 4.3. Let $A$ be a unital $C^*$-algebra, then $K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\}$, and $[0]_0 = 0$.

Proof. Every element of $K_0(A)$ can be written as $[([p]_\mathcal{D}, [q]_\mathcal{D})]_G$ for some $p, q \in \mathcal{P}_\infty(A)$, and
\[
([([p]_\mathcal{D}, [q]_\mathcal{D})]_G = ([([p]_\mathcal{D}, 0)]_G + ([0, [q]_\mathcal{D})]_G
= ([([p]_\mathcal{D}, 0)]_G) - ([([q]_\mathcal{D}, 0)]_G).
\]

Also,
\[
[0]_0 = ([([0]_\mathcal{D}, 0)]_G = ([0, 0])_G = 0.
\]

Proposition 4.4. Let $A, B$ and $C$ be $C^*$-algebras, let $\varphi : A \to B$ and $\psi : B \to C$ be continuous $*$-homomorphisms. Then $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi)$. Also, let $0$ denote the zero map between any two $C^*$-algebras, then $K_0(0) = 0$, the zero group map.

Proof. By Proposition 4.3, every element in $K_0(A)$ is of the form $[p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_\infty(A)$. Computing using Proposition 4.2,
\[
K_0(\psi) \circ K_0(\varphi)([p]_0 - [q]_0) = K_0(\psi)(K_0(\varphi)([p]_0) - K_0(\varphi)([q]_0))
= K_0(\psi)([\varphi(p)]_0 - [\varphi(q)]_0)
= [\psi \circ \varphi(p)]_0 - [\psi \circ \varphi(q)]_0
= K_0(\psi \circ \varphi)([p]_0 - [q]_0).
\]
Moreover,

\[ K_0(0)([p]_0 - [q]_0) = [0(p)]_0 - [0(q)]_0 = 0 - 0 = 0. \]

**Corollary 4.5.** The map \( K_0 \) is a (covariant) functor, with \( K_0 \) on \( C^* \)-algebras defined as in Definition 2.17 and \( K_0 \) on continuous \(*\)-morphisms defined as in Proposition 4.2.

**Proof.** Simply collect the results from Propositions 4.2 and 4.4. \( \blacksquare \)

## 5 \( K_0 \) of general \( C^* \)-algebras

Let \( A \) be a \( C^* \)-algebra, possibly non-unital. Let \( \tilde{A} \) denote the unitization of \( A \). Then \( \tilde{A} = A \oplus \mathbb{C}I \) as a vector space, and \( A \) is an ideal in \( \tilde{A} \). Let \( \iota_I, \iota_A \) be the inclusion maps from \( \mathbb{C}I \) and \( A \) into \( \tilde{A} \) respectively, and let \( \pi_I \) and \( \pi_A \) be the natural quotient maps from \( \tilde{A} \) onto \( \mathbb{C}I \) and \( A \) respectively. Both \( \tilde{A} \) and \( \mathbb{C}I \) are unital \( C^* \)-algebras. Their \( K_0 \) groups are defined as in the first section. Also, the inclusion \( \iota_I \) induces a group homomorphism \( K_0(\iota_I) : K_0(\mathbb{C}I) = \mathbb{Z} \to \tilde{A} \).

**Definition 5.1.** Let \( A \) be a \( C^* \)-algebra. Define \( K_0(A) = \ker K_0(\pi_I) \).

**Proposition 5.2.** Let \( A \) be a \( C^* \)-algebra. Then

\[
K_0(A) = \{ [p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(\tilde{A}), \pi_I(p) \sim_0 \pi_I(q) \} =: S_1
\]

\[
= \{ ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) : p, q \in \mathcal{P}_\infty(\tilde{A}) \} =: S_2
\]

\[
= \{ [p]_0 - [\pi_I(p)]_0 : p \in \mathcal{P}_\infty(\tilde{A}) \} =: S_3
\]

**Proof.** Let \( g \in K_0(\tilde{A}) \) and \( g \in \ker K_0(\pi_I) \). Then there exists some \( n \in \mathbb{N} \) and \( p, q \in \mathcal{P}_n(\tilde{A}) \) such that \( g = [p]_0 - [q]_0 \), and that

\[
0 = K_0(\pi_I)([p]_0 - [q]_0) = [\pi_I(p)]_0 - [\pi_I(q)]_0.
\]

So \( \pi_I(p) \sim_0 \pi_I(q) \). Conversely suppose \( \pi_I(p) \sim_0 \pi_I(q) \), then

\[
K_0(\pi_I)([p]_0 - [q]_0) = [\pi_I(p)]_0 - [\pi_I(q)]_0.
\]

This proves the first equality.
With the first equality in mind, suppose \( \pi_I(p) \sim_0 \pi_I(q) \). Then
\[
[p]_0 - [q]_0 = ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \in S_2.
\]
So \( \overline{K}_0(A) = S_1 \subseteq S_2 \). And
\[
K_0(\pi_I) \left( ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \right) \\
= ([\pi_I(p)]_0 - [\pi_I(q)]_0) - ([\pi_I \circ \pi_I(p)]_0 - [\pi_I \circ \pi_I(q)]_0) \\
= ([\pi_I(p)]_0 - [\pi_I(q)]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \\
= 0
\]
So \( S_2 \subseteq \overline{K}_0(A) \), this proves the second equality.

Clearly \( S_3 \subseteq S_2 \). Take
\[
g = ([p]_0 - [q]_0) - ([\pi_I(p)]_0 - [\pi_I(q)]_0) \in S_2.
\]
Suppose \( q \) is \( n \times n \), and let \( p' = p \oplus (I_n - q) \). Then
\[
[p']_0 = [p]_0 - [q_0] + [I_n]_0.
\]
Also
\[
\pi_I(p') = \pi_I(p) \oplus (I_n - \pi_I(q)),
\]
so
\[
[\pi_I(p')]_0 = [\pi_I(p)]_0 - [\pi_I(q)]_0 + [I_n]_n.
\]
Thus \( [p']_0 - [\pi_I(p)]_0 = g \), this proves \( S_2 = S_3 \). \( \blacksquare \)

The above gives a definition for the \( K_0 \) group of non-unital C*-algebras, and defines another abelian group for a unital C*-algebra. We need to verify that it coincides with the previous definition for the unital case.

**Lemma 5.3.** Let \( A \) be a unital C*-algebra. Let \( 1_A \) denote the identity of \( A \), and let \( \tilde{A} = A \oplus \mathbb{C}I \) as vector space. Then \( \tilde{A} \cong A \oplus \mathbb{C} J \). The C*-algebra \( A \oplus \mathbb{C} J \) is defined with norm \( \|a + zJ\| = \max(\|a\|, |z|) \) and involution \( (a + zJ)^* = a^* + \overline{z}J \).
Proof. Define $\tau : A \oplus \mathbb{C}J \to \widetilde{A}$ by $a \oplus zJ \mapsto a + z(I - 1_A)$. This is clear a vector space isomorphism and respects the involution. Lastly,

$$\tau(a \oplus zJ)\tau(b \oplus wJ) = (a + z(I - 1_A))(b + w(I - 1_A))$$
$$= ab + w(aI - a1_A) + z(Ib - 1_Ab) + zw(II - I1_A - 1_AI + 1_A1_A)$$
$$= ab + w(a - a) + z(b - b) + zw(I - 1_A - 1_A + 1_A)$$
$$= ab + zw(I - 1_A)$$
$$= \tau(ab \oplus zwJ).$$

So $\tau$ is an isomorphism. 

Remark 5.4. To gain an intuitive idea of the above lemma, consider the case of where $A = C(X)$ is the set of continuous functions from a compact Hausdorff space $X$ into the complex numbers. The unitization $\widetilde{C}(X)$ is isomorphic to $C(X \sqcup \{*\})$ (see Proposition 9.9). Let $1_A$ denote the function that is constantly 1 on $X$ and zero on $*$. Let $1_*$ be the function that is 1 on $*$ and constantly zero on $X$. Then we have

$$C(X \sqcup \{*\}) \cong C(X) \oplus C(\{*\}) \cong C(X) \oplus \mathbb{C}1_*,$$

where $1_* = 1 - 1_A$. The proof of the lemma imitates this idea to prove it in the non-commutative case.

Proposition 5.5. Let $A$ be a unital $C^*$-algebra, then $\overline{K}_0(A) \cong K_0(A)$.

Proof. By the lemma above, $\widetilde{A} \cong A \oplus \mathbb{C}J$. Let $\iota_A : A \to A \oplus \mathbb{C}J$ be the natural inclusion map and $\pi_A : A \oplus \mathbb{C}J \to A$ the quotient map. The map $\tau : A \oplus \mathbb{C}J \to \widetilde{A}$ is defined in the previous proof. Define $\alpha : K_0(A) \to K_0(\widetilde{A})$ by

$$[p]_0 - [q]_0 \mapsto [\tau(\iota_A(p))]_0 - [\tau(\iota_A(q))]_0.$$ 

In other words, $\alpha = K_0(\tau \circ \iota_A)$. Since $\pi_I(\tau(\iota_A(p))) = 0 = \pi_I(\tau(\iota_A(q)))$, the image of $\alpha$ is indeed in $\overline{K}_0(A)$. Let $\beta = K_0(\pi_A \circ \tau^{-1}) : \overline{K}_0(A) \to K_0(A)$. Then,

$$\beta \circ \alpha = K_0(\pi_A \circ \tau^{-1}) \circ \alpha = K_0(\pi_A \circ \iota_A) = K_0(\text{id}_A) = \text{id}_{K_0(A)}.$$
For \( \tilde{p}, \tilde{q} \in \mathcal{P}_\infty(\tilde{A}) \) with \( \pi_I(\tilde{p}) = \pi_I(\tilde{q}) \), let \( p_1 = \tau \circ \iota_A \circ \pi_A \circ \tau^{-1}(\tilde{p}) \) and \( p_2 = \tilde{p} - p_1 \). Then \( p_1 + p_2 = \tilde{p} \) and \( p_1, p_2 \) are orthogonal projections. Write \( \tilde{q} = q_1 + q_2 \) in the same way. Since \( \pi_I(\tilde{p}) = \pi_I(\tilde{q}) \), by the way that \( \tau \) is defined, we have that \( p_2 = q_2 \). So

\[
[p]_0 - [q]_0 = ([p_1]_0 + [p_2]_0) - ([q_1]_0 + [q_2]_0) = [p_1]_0 - [q_1]_0,
\]

and

\[
(\alpha \circ \beta)([p]_0 - [q]_0) = K_0(\tau \circ \iota_A \circ \pi_A \circ \tau^{-1})([p]_0 - [q]_0)
= [p_1]_0 - [q_1]_0 = [\tilde{p}]_0 - [\tilde{q}]_0.
\]

Hence \( \alpha \) and \( \beta \) are mutual inverses. \( \blacksquare \)

**Definition 5.6.** Let \( A \) be a non-unital C*-algebra. Define \( K_0(A) := \overline{K}_0(A) \).

**Remark 5.7.** By Proposition 5.5, we can safely write \( K_0(A) = \overline{K}_0(A) \) for any unital C*-algebras \( A \).

The description \( S_3 \) in Proposition 5.2 is the one will be used most often. Next is a discussion of when two elements in such description are equivalent.

**Lemma 5.8.** Let \( A \) be a C*-algebra, \( v \in M_{m,n}(A) \) and \( w \in M_{n,k}(A) \) for some \( k, m, n \in \mathbb{N} \). Then \( \pi_I(vw) = \pi_I(v)\pi_I(w) \).

**Proof.** We compute \( \pi_I(vw) \) to be

\[
\pi_I[(v - \pi_I(v))(w - \pi_I(w)) + \pi_I(v)(w - \pi_I(w)) + (v - \pi_I(v))w + \pi_I(v)\pi_I(w)]
\]

Since \( A \) is an ideal in \( \tilde{A} \), all of \( (v - \pi_I(v))(w - \pi_I(w)), \pi_I(v)(w - \pi_I(w)) \) and \( (v - \pi_I(v))w \) have entries in \( A \), which are 0 when they are evaluated under \( \pi_I \). So

\[
\pi_I(vw) = \pi_I(\pi_I(v)\pi_I(w)) = \pi_I(v)\pi_I(w)
\]

since \( \pi_I(v)\pi_I(w) \in M_{k,l}(\mathbb{C}I) \). \( \blacksquare \)

**Lemma 5.9.** Let \( A \) be a C*-algebra, and let \( p, q \in \mathcal{P}_\infty(\tilde{A}) \). Then \( p \sim_0 q \) in \( \mathcal{P}_\infty(\tilde{A}) \) implies \( \pi_I(p) \sim_0 \pi_I(q) \).
Proof. There exists a matrix \( v \) with entries in \( \widetilde{A} \) such that \( vv^* = p \) and \( v^*v = q \). By Lemma 5.8,

\[
\pi_I(p) = \pi_I(vv^*) = \pi_I(v)\pi_I(v^*) \sim_0 \pi_I(v^*)\pi_I(v) = \pi_I(v^*v) = \pi_I(q).
\]

Proposition 5.10. Let \( A \) be a C*-algebra, and \( p, q \in \mathcal{P}_\infty(\widetilde{A}) \). The following are equivalent

1. \([p]_0 - [\pi_I(p)]_0 = [q]_0 - [\pi_I(q)]_0\)
2. there exists \( r_1, r_2 \in \mathcal{P}_\infty(\widetilde{A}) \) with \( p \oplus r_1 \sim_0 q \oplus r_2 \)
3. there exists \( k, l \in \mathbb{N} \) such that \( p \oplus I_k \sim_0 q \oplus I_l \) in \( \mathcal{P}_\infty(\widetilde{A}) \)

Proof. (1 \( \implies \) 2) The equality \([p]_0 - [\pi_I(p)]_0 = [q]_0 - [\pi_I(q)]_0\) implies that

\[
[p \oplus \pi_I(q)]_0 = [p]_0 + [\pi_I(q)]_0 = [q]_0 + [\pi_I(p)]_0 = [q \oplus \pi_I(p)]_0
\]

So let \( r_1 = \pi_I(q) \) and \( r_2 = \pi_I(p) \). This satisfies 2.

(2 \( \implies \) 3) Since \( r_i = \pi_I(r_i) \) for \( i = 1, 2 \), we see that \( r_1 \) and \( r_2 \) can be considered as matrices in \( M_n(\mathbb{C}) \) and \( M_m(\mathbb{C}) \) respectively. Let \( k = \text{rank } r_1 \leq n \). Let \( \{z_1, \ldots, z_k\} \) be an orthonormal basis of \( \text{Ran} r_1 \mathbb{C}^n \), and extend it to an orthonormal basis \( \{z_1, \ldots, z_n\} \) of \( \mathbb{C}^n \). Let \( \{e_1, \ldots, e_n\} \) denote the standard basis of \( \mathbb{C}^n \), and define \( u \in M_n(\mathbb{C}) \) by \( uz_j = e_j \) for \( j = 1, \ldots, n \). Then \( u \) is unitary since it takes an orthonormal basis to another one, and

\[
ur_1 u^* e_j = ur_1 z_j = \begin{cases} 
uz_j = e_j & : j = 1, \ldots, k \\
u0 = 0 & : j = k + 1, \ldots, n
\end{cases}
\]

So

\[
r_1 \sim_0 ur_1 u^* = I_k \oplus 0_{n-k} \sim_0 I_k.
\]

By identifying \( u \) as a unitary matrix in \( M_k(\mathbb{C})I \), this also holds true in \( \mathcal{P}_\infty(\widetilde{A}) \).

Similarly, \( r_2 \sim_0 I_l \) in \( \mathcal{P}_\infty(\widetilde{A}) \) for \( l = \text{rank } r_2 \). So

\[
p \oplus I_k \sim_0 p \oplus r_1 \sim_0 q \oplus r_2 \sim_0 q \oplus I_l.
\]

(3 \( \implies \) 1) We use Lemma 5.9 here and compute

\[
[p]_0 - [\pi_I(p)]_0 = [p]_0 - [\pi_I(p)]_0 + [I_k]_0 - [I_k]_0
\]

\[
= [p \oplus I_k]_0 - [\pi_I(p) \oplus I_k]_0
\]

\[
= [p \oplus I_k]_0 - [\pi_I(p) \oplus I_k]_0
\]

\[
= [q \oplus I_l]_0 - [\pi_I(q) \oplus I_l]_0
\]

\[
= [q]_0 - [\pi_I(q)]_0.
\]
The next natural step is to extend the functor $K_0$ to all $\ast$-homomorphisms on all $C^\ast$-algebras. Let $A, B$ be $C^\ast$-algebras. A $\ast$-homomorphism $\varphi : A \to B$ can be extended to a $\ast$-homomorphism $\tilde{A} = A \oplus \mathbb{C}I_A \to \tilde{B} = B \oplus \mathbb{C}I_B$ by $\tilde{\varphi}|_A = \varphi$ and $\tilde{\varphi}(I_A) = I_B$.

**Definition 5.11.** Let $A, B$ be $C^\ast$-algebras, $\varphi : A \to B$ a $\ast$-homomorphism. Define $\overline{K}_0(\varphi) = K_0(\tilde{\varphi})|_{K_0(A)} : K_0(A) \to K_0(B)$. Then $\overline{K}_0(\varphi)$ is a well-defined group homomorphism.

**Proof.** Note that $\overline{K}_0(\varphi)$ is the restriction of $K_0(\tilde{\varphi})$ to $K_0(A)$. So it is a group homomorphism. $\pi_f(\tilde{\varphi}(p)) = \pi_f(\tilde{\varphi}(q))$ by the way $\tilde{\varphi}$ is defined. So the image of $\overline{K}_0(\varphi)$ is in $K_0(B)$. ■

**Proposition 5.12.** Let $A, B$ be unital $C^\ast$-algebras, let $\alpha : K_0(A) \to \overline{K}_0(A)$ be the group isomorphism described in the proof of Proposition 5.5, and similarly let $\beta : K_0(B) \to \overline{K}_0(B)$ be such group isomorphism. Then for any group homomorphism $\varphi : A \to B$, we have

$$\overline{K}_0(\varphi) \circ \alpha = \beta \circ K_0(\varphi).$$

**Proof.** We adopt all notation used in Proposition 5.5, where $\alpha = K_0(\tau_A \circ \iota_A)$ and $\beta = K_0(\tau_B \circ \iota_A)$. Then

$$\beta \circ K_0(\varphi) = K_0(\tau_B \circ \iota_A) \circ K_0(\varphi) = K_0(\tau_B \circ \iota_B \circ \varphi)$$

and

$$\overline{K}_0(\varphi) \circ \alpha = K_0(\tilde{\varphi})|_{\overline{K}_0(A)} \circ K_0(\tau_A \circ \iota_A) = K_0(\tilde{\varphi} \circ \tau_A \circ \iota_A).$$

For $a \in A$,

$$\tau_B \circ \iota_B \circ \varphi(a) = \varphi(a) \oplus 0I_B = \tilde{\varphi} \circ \tau_A \circ \iota_A(a).$$

So $\tau_B \circ \iota_B \circ \varphi = \tilde{\varphi} \circ \tau_A \circ \iota_A$ as maps $A \to \tilde{B}$, so applying $K_0$ they are the same as maps from $K_0(A)$ to $K_0(\tilde{B})$ whose image lie in $\overline{K}_0(B)$. This concludes the proof. ■

**Remark 5.13.** By the above proposition and Proposition 5.5, we can safely write $\overline{K}_0(\varphi) = K_0(\varphi)$ for any $\ast$-homomorphism $\varphi$.

**Proposition 5.14.** Let $A, B, C$ be $C^\ast$-algebras, and let $\varphi : A \to B$ and $\psi : B \to C$ be $\ast$-homomorphisms. Then $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi)$. Also, $K_0(\text{id}_A) = \text{id}_{K_0(A)}$ and $K_0(0) = 0$ for any zero map.
Proof. We compute:

\[
K_0(\psi) \circ K_0(\varphi) = K_0(\tilde{\psi})|_{K_0(B)} \circ K_0(\tilde{\varphi})|_{K_0(A)}
\]
\[
= K_0(\tilde{\psi} \circ \tilde{\varphi})|_{K_0(A)}
\]
\[
= K_0(\psi \circ \varphi)|_{K_0(A)}
\]
\[
= K_0(\psi \circ \varphi).
\]

Similarly,

\[
K_0(\text{id}_A) = K_0(\tilde{\text{id}}_A)|_{K_0(A)}
\]
\[
= K_0(\tilde{\text{id}}_A)|_{K_0(A)}
\]
\[
= \text{id}_{K_0(\tilde{A})}|_{K_0(A)}
\]
\[
= \text{id}_{K_0(A)}.
\]

Finally,

\[
K_0(0) = K_0(\tilde{0})|_{K_0(A)} = K_0(\pi_I)|_{K_0(A)}.
\]

But \(K_0(A)\) is exactly \(\text{ker} \ K_0(\pi_I)\), so \(K_0(0) = 0\).

Now we have a functor \(K_0\) from the category of \(C^*\)-algebras to the category of abelian groups.

6 Functorial properties of \(K_0\)

The \(K_0\)-group of a \(C^*\)-algebra can be difficult to compute even for most \(C^*\)-algebras. With the functoriality of \(K_0\) in hand, some useful properties of the functor \(K_0\) will aid calculation. One might say this is similar to how exact sequences help the computation of cohomology groups. In fact, \(K_0\) is an extraordinary cohomology functor, but this will not be discussed here. In short summary, the most basic and important properties of the functor \(K_0\) are homotopy invariance, half exactness and split exactness. Also, \(K_0\) is a continuous functor, meaning that the inductive limit \(K_0\)-group is isomorphic to the \(K_0\)-group of inductive limits. Other useful tools for computing the \(K_0\)-groups include the higher \(K\)-groups, Bott periodicity, and the 6-term exact sequence. In this paper we will only prove the three basic functorial properties of \(K_0\).
Definition 6.1. Let $A$ and $B$ be C*-algebras and $\varphi, \psi : A \to B$ be *-homomorphisms. We say $\varphi$ is homotopic to $\psi$, written $\varphi \sim_h \psi$, if there exists a family of continuous *-homomorphisms $\varphi_t : A \to B$ for $t \in [0,1]$ such that $\varphi_0 = \varphi$ and $\varphi_1 = \psi$, and that for each $a \in A$, $t \mapsto \varphi_t(a)$ is a continuous map $[0,1] \to B$. The family $\varphi_t$ is called a homotopy from $\varphi$ to $\psi$.

Let $A$ and $B$ be C*-algebras. We say $A$ is homotopic to $B$, written $A \sim_h B$, if there exists $\varphi : A \to B$ and $\psi : B \to A$ continuous *-homomorphisms such that $\varphi \circ \psi \sim_h \text{id}_A$ and $\psi \circ \varphi \sim_h \text{id}_B$.

6.1 Homotopy invariance

Proposition 6.2. Let $A$ and $B$ be C*-algebras, $\varphi, \psi : A \to B$ be continuous *-homomorphisms with $\varphi \sim_h \psi$, then $K_0(\varphi) = K_0(\psi)$. If $A \sim_h B$, then $K_0(A) \cong K_0(B)$.

Proof. Once again, a typical element in $K_0(A)$ is $[p]_0 - [q]_0$ for some $p, q \in \mathcal{P}_\infty(A)$. Hence it is sufficient to show that $K_0(\varphi)(p) = K_0(\psi)(p)$ for all $p \in \mathcal{P}_\infty$. Let $\varphi_t$ be a homotopy from $\varphi$ to $\psi$. The family $\varphi_t$ extends to a homotopy from $\varphi$ to $\psi$ on $M_n(A)$. The map $[0,1] \to M_n(B)$ given by $t \mapsto \varphi_t(p)$ is continuous, and since each $\varphi_t$ is a *-homomorphism, $\varphi_t(p) \in \mathcal{P}_n(B)$, so $t \mapsto \varphi_t(p)$ is a homotopy of

$$\varphi(p) = \varphi_0(p) \sim_h \varphi_1(p) = \psi(p).$$

But we know homotopic projections are equivalent in $\mathcal{D}(A)$, so

$$K_0(\varphi)(p) = [\varphi(p)]_0 = [\psi(p)]_0 = K_0(\psi)(p).$$

Hence $K_0(\varphi) = K_0(\psi)$.

Suppose $A \sim_h B$. There exists continuous homomorphisms $\alpha : A \to B$ and $\beta : B \to A$ such that $\alpha \circ \beta \sim_h \text{id}_A$ and $\beta \circ \alpha \sim_h \text{id}_B$. Then using Proposition 4.4 and the first half of this proof,

$$K_0(\alpha) \circ K_0(\beta) = K_0(\alpha \circ \beta) = K_0(\text{id}_A) = \text{id}_{K_0(A)},$$

$$K_0(\beta) \circ K_0(\alpha) = K_0(\beta \circ \alpha) = K_0(\text{id}_B) = \text{id}_{K_0(B)}.$$ 

Hence $K_0(\alpha) : K_0(A) \to K_0(B)$ is a group isomorphism, whose inverse is $K_0(\beta)$. ■

26
6.2 Half- and split-exactness

Definition 6.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor.

1. $\mathcal{F}$ is exact if whenever

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$$

is a short exact sequence in $\mathcal{C}$, then

$$0 \longrightarrow \mathcal{F}(A) \overset{\mathcal{F}(f)}{\longrightarrow} \mathcal{F}(B) \overset{\mathcal{F}(g)}{\longrightarrow} \mathcal{F}(C) \longrightarrow 0$$

is exact in $\mathcal{D}$.

2. $\mathcal{F}$ is half exact if whenever

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$$

is a short sequence in $\mathcal{C}$, then

$$\mathcal{F}(A) \overset{\mathcal{F}(f)}{\longrightarrow} \mathcal{F}(B) \overset{\mathcal{F}(g)}{\longrightarrow} \mathcal{F}(C)$$

is sequence in $\mathcal{D}$ that is exact at $\mathcal{F}(B)$.

3. $\mathcal{F}$ is split exact if whenever

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \overset{h}{\longleftarrow} 0$$

is a split exact sequence in $\mathcal{C}$, then

$$0 \longrightarrow \mathcal{F}(A) \overset{\mathcal{F}(f)}{\longrightarrow} \mathcal{F}(B) \overset{\mathcal{F}(g)}{\longrightarrow} \mathcal{F}(C) \overset{\mathcal{F}(h)}{\longleftarrow} 0$$

is a split exact sequence in $\mathcal{D}$.

Clearly an exact functor would be half-exact. In this section we will show that the functor $K_0$ is half-exact and split-exact. However, $K_0$ is not a exact functor. We will see a counterexample in a later section when we have developed more machinery.
Lemma 6.4. Let

$$0 \longrightarrow A \overset{\varphi}{\longrightarrow} B \overset{\psi}{\longrightarrow} C \longrightarrow 0$$

be a short exact sequence of C*-algebras, and let $n \in \mathbb{N}$. Let $\tilde{\varphi} : M_n(\tilde{A}) \to M_n(\tilde{B})$ and $\tilde{\psi} : M_n(\tilde{B}) \to M_n(\tilde{C})$ be the unital $\ast$-homomorphisms induced by $\varphi$ and $\psi$, respectively. Then,

1. The map $\tilde{\varphi} : M_n(\tilde{A}) \to M_n(\tilde{B})$ is injective.

2. An element $a \in M_n(\tilde{B})$ belongs to the image of $\tilde{\varphi}$ if and only if $\tilde{\psi}(a) = \pi_I(\tilde{\psi}(a))$.

Proof. 1. The map $\tilde{\varphi} : A \oplus CI_A \to B \oplus CI_B$ is injective on both $A$ and $CI_A$. Therefore it is injective $\tilde{A} \to \tilde{B}$, and also the induced map $\tilde{\varphi} : M_n(\tilde{A}) \to M_n(\tilde{B})$ is continuous.

2. For $a \in A$ and $z \in \mathbb{C}$,

$$\tilde{\psi} \circ \tilde{\varphi}(a + zI_A) = \tilde{\psi}(\varphi(a) + zI_B) = \psi \circ \varphi(a) + zI_C = zI_C$$

$$= \pi_I(\tilde{\psi} \circ \tilde{\varphi}(a + zI_A)).$$

Conversely, suppose $b \in B$ and $z \in \mathbb{C}$ with

$$\psi(b) + zI_C = \tilde{\psi}(b + zI_B) = \pi_I(\tilde{\psi}(b + zI_B)) = zI_C.$$

Then $\psi(b) = 0$. By exactness there exists $a \in A$ such that $\varphi(a) = b$, then $b + zI_B = \tilde{\varphi}(a + zI_A)$. \(\blacksquare\)

Proposition 6.5. $K_0$ is half-exact.

Proof. Let $A, B$ and $C$ be C*-algebras with $\ast$-homomorphisms $\varphi : A \to B$ and $\psi : B \to C$, where $\varphi$ is injective, $\psi$ is surjective, and $\text{im}(\varphi) = \ker(\psi)$.

A typical element in $K_0(A)$ is $[p]_0 - [\pi_I(p)]_0$ for some $p \in \mathcal{P}_\infty(A)$. By Lemma 6.4 the equation

$$\tilde{\psi} \circ \tilde{\varphi}(p) = \pi_I(\tilde{\psi} \circ \tilde{\varphi}(p)) = \tilde{\psi} \circ \tilde{\varphi}(\pi_I(p))$$

holds. So

$$K_0(\psi) \circ K_0(\varphi)([p]_0 - [\pi(p)]_0) = [\tilde{\psi} \circ \tilde{\varphi}(p)]_0 - [\tilde{\psi} \circ \tilde{\varphi}(\pi_I(p))]_0 = 0.$$
So \( \text{im}(K_0(\varphi)) \subseteq \ker(K_0(\psi)) \).

Conversely, let \([p]_0 - [\pi_I(p)]_0 \in K_0(B)\) be in the kernel of \(K_0(\psi)\). Since \(\tilde{\psi}(p) \sim_0 \tilde{\psi}(\pi_I(p))\) in \(P_n(C)\) for some \(n \in \mathbb{N}\), by Proposition 3.15 there exists a unitary element \(u \in M_{2n}(C)\) such that
\[
u(\tilde{\psi}(p) \oplus 0_n)u^* = \tilde{\psi}(\pi_I(p)) \oplus 0_n.
\]

By Lemma 3.10 there exists a unitary \(v \in M_{4n}(B)\) such that \(\tilde{\psi}(v) = u \oplus u^*\). Let \(p_1 = v(p \oplus 0_{3n})v^*\). Then
\[
p \sim_0 p \oplus 0_{3n} \sim_0 p_1,
\]
and similarly \(\pi_I(p) \sim_0 \pi_I(p_1)\). Also,
\[
\begin{align*}
\tilde{\psi}(p_1) &= \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} \tilde{\psi}(p) \oplus 0_n & 0 \\ 0 & 0_{2n} \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & u \end{bmatrix} \\
&= \begin{bmatrix} u(\tilde{\psi}(p) \oplus 0_n)u^* & 0 \\ 0 & 0_{2n} \end{bmatrix} \\
&= \pi_I(\tilde{\psi}(p)) \oplus 0_{3n}.
\end{align*}
\]

It follows that \(\tilde{\psi}(p_1) = \pi_I(\tilde{\psi}(p_1))\). By Lemma 6.4 there exists \(e \in M_{3n}\) such that \(\tilde{\varphi}(e) = p_1\). Also,
\[
\tilde{\varphi}(ee) = \tilde{\varphi}(e)\tilde{\varphi}(e) = p_1p_1 = p_1,
\]
\[
\tilde{\varphi}(e^*) = p_1^* = p_1.
\]

By Lemma 6.4, \(\tilde{\varphi} : M_{4n}({\tilde{A}}) \to M_{4n}({\tilde{B}})\) is injective, which implies \(e = ee = e^*\), and hence \(e\) is a projection. Now
\[
K_0(\varphi)([e]_0 - [\pi_I(e)]_0) = [p_1]_0 - [\pi_I(p_1)]_0 = [p]_0 - [\pi_I(p)]_0.
\]

This shows that \(\ker K_0(\psi) \subseteq \ker K_0(\varphi)\). Therefore \(\ker K_0(\psi) = \text{im} K_0(\varphi)\). \(\blacksquare\)

**Proposition 6.6.** The functor \(K_0\) is split-exact.

**Proof.** Suppose
\[
0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0
\]
is a split exact sequence of C*-algebras. By the half-exactness just proved, the sequence
\[ K_0(A) \xrightarrow{K_0(\varphi)} K_0(B) \xrightarrow{K_0(\psi)} K_0(C) \]
is exact. Also, since \( K_0 \) is a functor, we have
\[ K_0(\psi) \circ K_0(\lambda) = K_0(\psi \circ \lambda) = K_0(\text{id}_C) = \text{id}_{K_0(C)}, \]
so the sequence is also exact at \( K_0(C) \). It is left to show that \( K_0(\varphi) \) is injective.

Let \( g \in K_0(A) \) be in the kernel of \( K_0(\varphi) \). By the proof of Proposition 6.5, there exits some \( n \in \mathbb{N} \), \( p \in \mathcal{P}_n(\tilde{A}) \) and some unitary \( u \in M_n(\tilde{B}) \) such that \( g = [p]_0 - [\pi_I(p)]_0 \) and \( u\tilde{\varphi}(p)u^* = \pi_I(\tilde{\varphi}(p)) \). Let \( v = (\tilde{\lambda} \circ \tilde{\psi})(u^*)u \). Then
\[ v^*v = u^*(\tilde{\lambda} \circ \tilde{\psi}(u))(\tilde{\lambda} \circ \tilde{\psi}(u^*))u = u^*I_n u = I_n, \]
\[ vv^* = (\tilde{\lambda} \circ \tilde{\psi}(u^*))uu^*(\tilde{\lambda} \circ \tilde{\psi}(u)) = I_n, \]
and
\[ \tilde{\psi}(v) = (\tilde{\psi} \circ \tilde{\lambda} \circ \tilde{\psi}(u^*))(\tilde{\psi}(u)) = \tilde{\psi}(u^*)\tilde{\psi}(u) = \tilde{\psi}(I_n) = I_n. \]
Since \( \tilde{\psi}(v) = \pi_I(\tilde{\psi}(v)) \), by Lemma 6.4, there exists \( w \in M_n(\tilde{A}) \) such that \( \tilde{\varphi}(w) = v \). Since \( \tilde{\varphi} \) is injective and \( \tilde{\varphi}(w^*w) = I_n = \tilde{\varphi}(ww^*) \), have \( ww^* = I_n = w^*w \), so \( w \) is unitary. Moreover,
\[ \tilde{\varphi}(wpw^*) = v\tilde{\varphi}(p)v^* = (\tilde{\lambda} \circ \tilde{\psi})(u^*)u\tilde{\varphi}(p)u^*(\tilde{\lambda} \circ \tilde{\psi})(u) \]
\[ = (\tilde{\lambda} \circ \tilde{\psi})(u^*)\pi_I(\tilde{\varphi}(p))(\tilde{\lambda} \circ \tilde{\psi})(u) \]
\[ = (\tilde{\lambda} \circ \tilde{\psi})(u^*\pi_I(\tilde{\varphi}(p)))u \]
\[ = (\lambda \circ \tilde{\psi})(\tilde{\varphi}(p)) = \tilde{\lambda}((\tilde{\psi} \circ \tilde{\varphi})(p)) \]
\[ = \tilde{\lambda}((\tilde{\psi} \circ \tilde{\varphi})(\pi_I(p))) \]
\[ = \tilde{\varphi}(\pi_I(p)). \]
By the injectivity of \( \tilde{\varphi} \) we can conclude that \( \pi_I(p) = wpw^* \). Hence \( p \sim_0 \pi_I(p) \) in \( \mathcal{P}_n(\tilde{A}) \). Therefore \( g = 0. \)

**Corollary 6.7.** Let \( A \) and \( B \) be C*-algebras. Then \( K_0(A \oplus B) \cong K_0(A) \oplus K_0(B) \).
Proof. The sequence
\[ 0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0 \]
is split-exact. Hence by the split-exactness of \( K_0 \), we have a split-exact sequence of abelian groups:
\[ 0 \longrightarrow K_0(A) \longrightarrow K_0(A \oplus B) \longrightarrow K_0(B) \longrightarrow 0. \]
Therefore \( K_0(A) \oplus K_0(B) \cong K_0(A \oplus B) \).

## 7 K-theory of compact Hausdorff spaces

**Definition 7.1.** Let \( X \) be a Hausdorff topological space, \( V \) and \( W \) topological vector bundles over \( X \). Define the map \( \pi_V : V \to X \) by \( \pi_V(v) = x \) if \( v \in V_x \). We write \( \pi = \pi_V \), when it is understood that \( \pi \) has domain \( V \). A map \( \varphi : V \to W \) is a bundle homomorphism if \( \varphi \) is continuous, \( \varphi(v) \in \pi_W^{-1}(\pi_V(v)) \) for all \( v \in V \), and that \( \varphi_x = \varphi|_{V_x} : V_x \to W_x \) is a linear homomorphism for all \( x \in X \). We say \( V \) is isomorphic to \( W \) if there exists \( \varphi : V \to W \) and \( \psi : W \to V \) bundle homomorphisms such that \( \varphi \circ \psi = \text{id}_V \) and \( \psi \circ \varphi = \text{id}_W \).

**Definition 7.2.** Let \( X \) be a Hausdorff space and let \( n \in \mathbb{N} \). Define \( \Theta^n(X) \) to be the rank-\( n \) trivial bundle over \( X \); specifically, \( \Theta^n(X) = X \times \mathbb{C}^n \).

**Definition 7.3.** For \( X \) a Hausdorff space, define \( \text{Vect}(X) \) to be the set of all isomorphism classes of topological vector bundles on \( X \).

**Definition 7.4.** Let \( X \) be a Hausdorff space, define \( C(X) \) to be the set of all continuous functions from \( X \) to \( \mathbb{C} \). If \( X \) is compact, then \( C(X) \) can be equipped with the sup-norm as the norm and with pointwise conjugation as its involution. This gives \( C(X) \) a \( C^* \)-algebra structure.

**Remark 7.5.** Let \( \mathcal{C} \) be the category of compact Hausdorff spaces and let \( \mathcal{A} \) be the category of unital \( C^* \)-algebras. Define a contravariant functor \( \mathcal{F} : \mathcal{C} \to \mathcal{A} \) as follows. If \( X \) is a compact Hausdorff space, then \( \mathcal{F}(X) = C(X) \). If \( X, Y \) are compact Hausdorff spaces and \( \varphi \in \text{Hom}(X, Y) \), then \( \mathcal{F}(\varphi) = \varphi^* \in \text{Hom}(C(Y), C(X)) \) where \( \varphi^* f(x) = f(\varphi(x)) \) for all \( f \in C(Y) \) and \( x \in X \), where \( \text{Hom}(X, Y) \) is the set of continuous functions from \( X \) to \( Y \), and \( \text{Hom}(C(Y), C(X)) \) is the set of \( * \)-homomorphisms from \( C(Y) \) to \( C(X) \).
If $X$ is a Hausdorff space, not necessarily compact, then $C(X)$ is not necessarily a C*-algebra since the sup-norm cannot be defined. However $C(X)$ is a ring, so for $m, n \in \mathbb{N}$, it makes sense to consider $M_{m,n}(C(X))$, all $m$ by $n$ matrices with entries in $C(X)$. Note that $M_{m,n}(C(X))$ is naturally isomorphic to $C(X, M_{m,n}(\mathbb{C}))$, by taking a matrix $F \in M_{m,n}(C(X))$ to $f \in C(X, M_{m,n}(\mathbb{C}))$, where $[f(x)]_{ij} = F_{ij}(x)$ for all $x \in X$.

**Lemma 7.6.** Let $X$ be a Hausdorff space, and let $m, n \in \mathbb{N}$. For every $f \in C(X, M_{m,n}(\mathbb{C}))$, define a bundle homomorphism $\Gamma(f) : \Theta^n(X) \to \Theta^m(X)$ by $\Gamma(f)(x, v) = (x, f(x)v)$. Then $\Gamma : f \mapsto \Gamma(f)$ is a bijection from $C(X, M_{m,n}(\mathbb{C}))$ to $\text{Hom}(\Theta^n(X), \Theta^m(X))$. In other words, we have a one-to-one correspondence between $\text{Hom}(\Theta^n(X), \Theta^m(X))$ and $C(X, M_{m,n}(\mathbb{C})) = M_{m,n}(C(X))$.

**Proof.** Suppose $f, g \in M_{m,n}(C(X))$ with $f \neq g$. Pick $x \in X$ for which $f(x) \neq g(x)$. Then there exists $v \in \mathbb{C}^n$ for which $g(x)v \neq f(x)v$, which shows that $\Gamma$ is injective. It is left to show that $\Gamma$ is surjective.

Let $\mathbb{C}^n$ and $\mathbb{C}^m$ be equipped with their standard inner products. Define $p : \Theta^n(X) \to \mathbb{C}^n$ by $p(x, w) = w$. Suppose $\varphi : \Theta^n(X) \to \Theta^m(X)$ is a bundle homomorphism. Define $f : X \to M_{m,n}(\mathbb{C})$ so that $f(x)_{ij} = \langle p(\varphi(x, e_j)), e_i \rangle$

for all $x \in X$. Clearly $f$ is continuous. Moreover,

$$\Gamma(f)(x, v) = (x, f(x)v)$$

$$= (x, \sum_{i=1}^m \sum_{j=1}^n f(x)_{ij}v_j e_i)$$

$$= (x, \sum_{i=1}^m \sum_{j=1}^n \langle p(\varphi(x, e_j)), e_i \rangle v_j e_i)$$

$$= (x, \sum_{i=1}^m \sum_{j=1}^n \langle p(\varphi(x, v_j e_j)), e_i \rangle e_i)$$

$$= (x, \sum_{i=1}^m \langle p(\varphi(x, v)), e_i \rangle e_i)$$

$$= (x, \varphi(x, v))$$

for all $(x, v) \in \Theta^n(X)$. Thus $\Gamma(f) = \varphi$, and we conclude that $\Gamma$ is surjective. ■
Lemma 7.7. Let $V$ and $W$ be vector bundles over a compact Hausdorff space $X$, and suppose that $\varphi : V \to W$ is a bundle homomorphism such that $\varphi_x$ is a vector space isomorphism for every $x \in X$. Then $\varphi$ is a bundle isomorphism.

Proof. Let $X_1, \ldots, X_k$ be the connected components of $X$, let $V_j = V|_{X_j}$ and $W_j = W|_{X_j}$ for $j = 1, \ldots, k$. If $\varphi : V \to W$ is a bundle homomorphism such that $\varphi|_{V_j}$ is an isomorphism from $V_j$ onto $W_j$, then $\varphi$ is an isomorphism from $V$ onto $W$. Thus for the rest of the proof we may assume that $X$ is connected.

By hypothesis $\varphi$ is a bijection, so $\varphi^{-1}$ is defined, with $\varphi^{-1}|_x$ a vector space isomorphism. We need to check that $\varphi^{-1}$ is continuous. Choose an open cover \{\{U_1, \ldots, U_l\} for which $V|_{U_k}$ and $W|_{U_k}$ are trivial for $k = 1, \ldots, l$. For each $k$, let $\varphi_k = \varphi|_{U_k}$. Then it is sufficient to show that $\varphi_k^{-1}$ is continuous.

Let $n$ be the rank of $V$ and $W$. We can identify $V|_{U_k}$ and $W|_{U_k}$ with $\Theta^n(U_k)$, and can consider $\varphi_k$ to be a bundle isomorphism from $\Theta^n(U_k)$ to itself. Apply Lemma 7.6 to obtain a continuous function $f_k : U_k \to M_n(\mathbb{C})$ such that $\varphi_k(x, v) = (x, f_k(x)v)$ for all $(x, v) \in \Theta^n(U_k)$. Since $\varphi_k(x)$ is an isomorphism for all $x \in U_k$, have $f_k(x) \in GL_n(\mathbb{C})$ for all $x \in U_k$.

Each $f_k$ is an element of $C(U_k, M_n(\mathbb{C}))$. The matrix $f_k(x)$ is invertible for every $x \in U_k$, since inversion is continuous, we have that $f^{-1}(x) \in C(U_k, M_n(\mathbb{C}))$. Apply the lemma again have $\varphi_k^{-1}$ is continuous. ■

Proposition 7.8. Let $V$ be a vector bundle over a compact Hausdorff space $X$. Then $V$ is isomorphic to a subbundle of the trivial bundle $\Theta^N(X)$ for some $N \in \mathbb{N}$.

Proof. Let $X_1, \ldots, X_m$ be the distinct connected components of $X$. If $V|_{X_k}$ is a subbundle of $\Theta^{N_k}(X_k)$ for some $N_k \in \mathbb{N}$, then let $N = N_1 + N_2 + \cdots + N_m$, and $V$ is itself a subbundle of $\Theta^N(X)$. So for the rest of the proof we may assume that $X$ is connected.

Since $V$ is locally trivial, let $\mathcal{U} = \{U_1, \ldots, U_l\}$ be an open cover of $X$ such that $V|_{U_k} \cong \Theta^M(U_k)$ for some $M \in \mathbb{N}$. (Note that this $M$ is the same for all $k$ since $X$ is connected.) Let $\varphi_k : V|_{U_k} \to \Theta^M(U_k)$ be a bundle isomorphism. Define $q_k : \Theta^M(U_k) \to \mathbb{C}^M$ by $q_k(x, w) = w$ for $x \in U_k$ and $w \in \mathbb{C}^M$; also let $\pi : V \to X$ be projection onto the point in $X$ that an element $v \in V$ lies above. Choose a partition of unity \{\{f_1, \ldots, f_l\} subordinate to the cover $\mathcal{U}$, and let $N = M \cdot l$. Then define $\Phi : V \to \bigoplus_{k=1}^l \mathbb{C}^M$ by

$$\Phi(v) = (f_1(\pi(v))q_1(\varphi_1(v)) \oplus \cdots \oplus f_l(\pi(v))q_l(\varphi_l(v))).$$
Then $\varphi(v) = (\pi(v), \Phi(v))$ defines a bundle homomorphism $V \to \Theta^N(X)$. Since $\varphi$ is injective, this is a bijective homomorphism onto a subbundle of $\Theta^N(X)$. By Lemma 7.7 this is indeed an isomorphism.

**Corollary 7.9.** Every vector bundle over a compact Hausdorff space admits a Hermitian metric.

*Proof.* It is clear that every trivial bundle naturally has a Hermitian metric, and since every bundle over a compact Hausdorff space is a subbundle of some trivial bundle, then it inherits the restriction of the Hermitian metric.

**Definition 7.10.** Let $X$ be a Hausdorff space, and let $[V], [W] \in \text{Vect}(X)$. Define $[V \oplus W]$ to be the isomorphism class of bundles as follows. There exists $n, m \in \mathbb{N}$ such that $V$ is a subbundle of $\Theta^n(X)$ and $W$ is a subbundle of $\Theta^m(X)$. Let $Q$ be the subbundle of $\Theta^{n+m}(X)$ such that $Q_x = V_x \oplus W_x \subseteq \mathbb{C}^n \oplus \mathbb{C}^m$ for all $x \in X$. Define $[V \oplus W]$ to be $[Q]$.

**Proposition 7.11.** Let $X$ be a compact Hausdorff space, and let $V, W$ be vector bundles over $X$. Then $[V \oplus W]$ is well-defined and it is a vector bundle.

*Proof.* The proof is easy and is left as an exercise for the reader.

**Remark 7.12.** The vector bundle $V \oplus W$ is called the Whitney sum of $V$ and $W$. The general construction is more abstract and it may take some work to check the bundle definitions. Proposition 7.8 allows for a concrete description of the class $[V \oplus W]$. Also, in K-theory it is more helpful to think of a vector bundle as a subbundle of some trivial bundle, as we will see when we relate the topological K-theory to the C*-algebra K-theory.

**Proposition 7.13.** Let $X$ be a compact Hausdorff space. The set $\text{Vect}(X)$ equipped with the operation $[V] + [W] = [V \oplus W]$, is an abelian monoid.

*Proof.* The only non-trivial part is to verify that $[V] + [W] = [W] + [V]$. Suppose $V$ is a subbundle of $\Theta^n(X)$ and $W$ is a subbundle of $\Theta^m(X)$. We’ll write $V \oplus W$ and $W \oplus V$ as the corresponding subbundles of $\Theta^{n+m}(X)$. Let $\rho : V \oplus W \to W \oplus V$ be such that

$$\rho(x, v \oplus w) = \rho(x, w \oplus v)$$
for all $x \in X$ and $v \in V_x$, $w \in W_x$. Clearly $\rho|_x$ is a vector space isomorphism for all $x \in X$, so by Lemma 7.7 it is left to show that $\rho$ is continuous. For any $x \in X$, take an open neighbourhood $U$ of $x$ for which both $V|_U$ and $W|_U$ are trivial. There exists $k \leq n$ and $l \leq m$ for which there exists bundle isomorphisms

$$\varphi : V|_U \xrightarrow{\cong} \Theta^k(U); \quad \psi : W|_U \xrightarrow{\cong} \Theta^l(U).$$

**Definition 7.14.** Let $X$ be a compact Hausdorff space. Define $K^0(X) = G(\text{Vect}(X))$, where $G(\cdot)$ is the Grothendieck completion.

The following is a lemma that helps with computation of $K^0$-groups.

**Lemma 7.15.** Let $X$ be a compact Hausdorff space and let $I$ denote the closed interval $[0,1]$. If $V$ is a vector bundle over $X \times I$, then $V|_{X \times \{0\}} \cong V|_{X \times \{1\}}$.

**Proof.** First we show that a bundle $V$ over $X \times [a, b]$ is trivial if there exists some $c \in (a, b)$ such that $V|_{X \times [a, c]}$ and $V|_{X \times [c, b]}$ are trivial. To see this, let $\varphi : V|_{X \times [a, c]} \to \Theta^n(X \times [a, c])$ and $\psi : V|_{X \times [c, b]} \to \Theta^n(X \times [c, b])$ be bundle isomorphisms for some $n \in \mathbb{N}$. There exists a function $h : X \to GL_n(\mathbb{C})$ such that $\varphi(v) = h(\pi(v))\psi(v)$ for all $v \in V|_x$. Then the map $\Phi : V \to \Theta^n(X \times [a, b])$ defined by

$$\Phi(v) = \begin{cases} \varphi(v) & : a \leq t \leq c \\ h(\pi(v))\psi(v) & : c < t \leq b \end{cases}$$

is a bundle isomorphism.

Next, for every $x \in X$ and $t \in [0,1]$ there exists some $U_{x,t} \subseteq X$ a neighbourhood of $x$ and some $\delta_t > 0$ such that $V$ is trivial over

$$U_{x,t} \times (t - \delta_t, t + \delta_t).$$

Because $[0,1]$ is compact, there exists a finite collection $\{t_0, \ldots, t_k\} \subseteq [0,1]$ such that

$$\bigcup_{i=0}^{k}(t_i - \delta_{t_i}, t_i + \delta_{t_i}) \supseteq [0,1].$$

Let $U_x = \bigcap_{i=0}^{k} U_{x,t_i}$. Then $V$ is trivial over $U_x \times (t_i - \delta_{t_i}, t_i + \delta_{t_i})$ for all $i = 0, \ldots, k$. Hence by observation from the previous paragraph, we see
that \( V|_{U \times I} \) is trivial. Thus, since \( X \) is compact, there exists a finite cover \( \{ U_1, \ldots, U_r \} \) of \( X \) such that \( V|_{U_j \times I} \) is trivial for all \( j = 1, \ldots, r \).

Let \( \{ f_1, \ldots, f_r \} \) be a partition of unity subordinate to the cover \( \{ U_1, \ldots, U_r \} \).

For \( j = 0, \ldots, r \) let

\[
F_j = f_1 + \cdots + f_j.
\]

In particular \( F_0 = 0 \) and \( F_r = 1 \). Also define

\[
X_0 = \{(x, F_j(x)) : x \in X \}
\]

for \( j = 1, \ldots, r \). Because \( V|_{U_j \times I} \) is trivial, there exists a bundle isomorphism

\[
\Phi_j : V|_{U_j \times I} \to \Theta^n(U_j \times I).
\]

Define \( \Psi_j : V|_{X_j - 1} \to V|_{X_j} \) by

\[
\Psi_j(v) = \begin{cases} v & : \pi(v) \notin U_j \times I \\ \Phi_j^{-1}(w) & : \pi(v) \in U_j \times I \end{cases}
\]

where \( w = ((x, f_j(x)), u) \) if \( \Phi_j(v) = ((x, f_{j-1}(x)), u) \). Then \( \Psi_j \) is a bundle isomorphism. Thus we have

\[
V|_{X \times \{0\}} = V|_{X_0} \cong V|_{X_1} \cong \ldots \cong V|_{X_r} = V|_{X \times \{1\}}.
\]

**Corollary 7.16.** Every vector bundle over a contractible compact Hausdorff space is trivial.

**Proof.** Let \( X \) be a contractible compact Hausdorff space. There exists a fixed point \( x_0 \in X \) and a continuous function \( \varphi : X \times [0, 1] \to X \) satisfying \( \varphi|_{X \times \{0\}}(x) = x \) for all \( x \in X \) and \( \varphi|_{X \times \{1\}}(x) = x_0 \) for all \( x \in X \). Suppose \( V \) is a vector bundle over \( X \). Then \( \varphi^*(V) \) is a bundle over \( X \times [0, 1] \) with

\[
V \cong \varphi^*(V)|_{X \times \{0\}} \cong \varphi^*(V)|_{X \times \{1\}} \cong \Theta^n V(X)
\]

by Lemma 7.15. \( \blacksquare \)

**Example 7.17.** Consider the compact Hausdorff space \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). Let \( A = \{ e^{i\theta} : 0 \leq \theta \leq \pi \} \) be the closed upper half of \( S^1 \) and let \( B = \{ e^{i\theta} : \pi \leq \theta \leq 2\pi \} \) be the lower half of \( S^1 \). Fix a rank \( n \) complex vector bundle \( V \) over \( S^1 \). Because \( A \) and \( B \) are both contractible, by Corollary 7.16 \( V|_A \) and \( V|_B \) are trivial bundles. Let \( \varphi : V|_A \to \Theta^n(A) \) and \( \psi : V|_B \to \Theta^n(B) \) be bundle isomorphisms. Let \( g \in GL_n(\mathbb{C}) \) be the matrix that represents \( \varphi \circ \psi^{-1} \) at 1, and let \( h \) be the matrix that represents \( \varphi \circ \psi^{-1} \) at \(-1\). The
The group $GL_n(\mathbb{C})$ is path connected, so let $g_t$ and $h_t$ be continuous paths from $A$ and $B$ respectively to the identity matrix.

Define a rank $n$ bundle $W$ over $S^1 \times I$ as follows. The bundle $W$ is trivial over $A \times I$ and $B \times I$, with trivializations $\Phi : W|_{A \times I} \to \Theta^n(A \times I)$ and $\Psi : W|_{B \times I} \to \Theta^n(B \times I)$. Furthermore, the transition function is defined to be

$$
\Psi^{-1}((1, t), u) = \Phi^{-1}((1, t), g_t u) \quad \text{and} \quad \Psi^{-1}((-1, t), u) = \Phi^{-1}((-1, t), h_t u)
$$

for $\pm 1 \in S^1$, $t \in [0, 1]$ and $u \in \mathbb{C}^n$. Finally, Lemma 7.15 implies that $V \cong W|_{S^1 \times \{0\}} \cong W|_{S^1 \times \{1\}} \cong \Theta^n(S^1)$.

Therefore equivalence classes of vector bundles over $S^1$ are characterized by ranks, and $K_0(S^1) \cong G(\mathbb{N}) \cong \mathbb{Z}$.

8 $K^0(X) \cong K_0(C(X))$

The main result of this section is the proof of the equivalence of $K$-theories. When $X$ is compact Hausdorff, then $C(X)$ is a unital $C^*$-algebra, and it makes sense to ask if the two definitions of $K$-theories agree.

**Theorem 8.1.** Let $X$ be compact Hausdorff. Then $K_0(C(X)) \cong K^0(X)$ as abelian groups.

Now we will develop some results necessary to prove this theorem.

**Definition 8.2.** Let $X$ be a compact Hausdorff space. For $E \in \mathcal{P}_\infty(C(X))$, and $x \in X$, let $\text{Ran} \ E(x)$ be the image of $E(x)$. That is, if $E$ is $n \times n$, then $\text{Ran} \ E(x) = E(x)\mathbb{C}^n$. Define $\text{Ran} \ E = \bigcup_{x \in X} \bigcup_{v \in \text{Ran} \ E(x)} (x, v)$.

**Proposition 8.3.** Let $X$ be a compact Hausdorff space, $n \in \mathbb{N}$ and $E \in \mathcal{P}_\infty(C(X))$. Then $\text{Ran} \ E$ is a vector bundle over $X$.

**Proof.** Fix $x_0 \in X$ and let

$$
U = \{x \in X : \|E(x_0) - E(x)\|_{op} < 1\}
$$

As $E$ and the operator norm are both continuous, the set $U$ is the pull back of $(-\infty, 1)$ through a continuous function, and is hence open. Observe that
for any \( x_1 \in X \), the element \( I_n + E(x_0) - E(x_1) \) is within distance 1 from \( I_n \), and as such is an invertible matrix. Also, for any \( v \in \mathbb{C}^n \), we have

\[
(I_n + E(x_0) - E(x_1))E(x_1)v = E(x_1)v + E(x_0)E(x_1)v - E(x_1)E(x_1)v \\
= E(x_1)v + E(x_0)E(x_1)v - E(x_1)v \\
= E(x_0)E(x_1)v
\]

So \( I_n + E(x_0) - E(x_1) \) maps \( \text{Ran} \, E(x_1) \) into \( \text{Ran} \, E(x_0) \), and since this is an invertible matrix, we have that \( \dim \text{Ran} \, E(x_0) \geq \dim \text{Ran} \, E(x_1) \). A similar calculation shows that

\[
(I_n - E(x_0) + E(x_1))(\text{Ran} \, E(x_0)) \subseteq \text{Ran} \, E(x_1)
\]

Thus we see that \( \text{Ran} \, E(x_0) \) and \( \text{Ran} \, E(x_1) \) have the same dimension, and \( I_n + E(x_0) - E(x_1) \) maps \( \text{Ran} \, E(x_1) \) to \( \text{Ran} \, E(x_0) \) isomorphically. Thus, the map

\[
\varphi : \text{Ran} \, E|_U \to U \times \text{Ran} \, E(x_0) \\
(x, v) \mapsto (x, (I_n + E(x_0) - E(x_1))v)
\]

is a bundle isomorphism. So \( \text{Ran} \, E \) is locally trivial, thus is a vector bundle. ■

**Proposition 8.4.** Let \( X \) be a compact Hausdorff space, and let \( E, F \in \mathcal{P}_\infty(C(X)) \). Then \( \text{Ran} \, E \cong \text{Ran} \, F \) as bundles if and only if \( E \sim_u F \).

**Proof.** Since \( \text{Ran} \, Q \cong \text{Ran} \, (\text{diag}(Q, 0_r)) \) for any \( Q \in \mathcal{P}_\infty(C(X)) \) and \( r \in \mathbb{N} \), we can take some \( n \in \mathbb{N} \) large enough so that \( E \) and \( F \) are both in \( M_n(C(X)) \).

Suppose that \( E \sim_u F \). Then we can find \( U \in \mathcal{U}_n(C(X)) \) such that \( UEU^* = F \). Define \( \gamma : \text{Ran} \, E \to \text{Ran} \, F \) by

\[
\gamma(x, E(x)v) = (x, U(x)E(x)v) = (x, F(x)U(x)v) \in \text{Ran} \, F(x),
\]

for \( x \in X \) and \( v \in \mathbb{C}^n \). It has the inverse map

\[
\gamma^{-1}(x, F(x)v) = (x, U^*(x)F(x)v) = (x, E(x)U^*(x)v).
\]

So \( \gamma \) is a bundle isomorphism between \( \text{Ran} \, E \) and \( \text{Ran} \, F \).

Conversely, suppose that \( \text{Ran} \, E \) and \( \text{Ran} \, F \) are isomorphic vector bundles. Let \( \varphi : \text{Ran} \, E \to \text{Ran} \, F \) be a bundle isomorphism. We define matrices
$A, B \in M_n(C(X))$ as follows. For $f \in (C(X))^n$, let $Af = \varphi(Ef)$ and $Bf = \varphi^{-1}(Ff)$. Then

$$ABf = A(\varphi^{-1}(Ff)) = \varphi(E(\varphi^{-1}(Ff))).$$

However $\varphi^{-1}(Ff)$ is a continuous section of $\text{Ran} E$, so

$$ABf = \varphi(E(\varphi^{-1}(Ff))) = \varphi(\varphi^{-1}(Ff)) = Ff$$

Which shows that $AB = F$. A similar computation shows that $BA = E$. Also,

$$EBf = E\varphi^{-1}(Ff) = \varphi^{-1}(Ff) = Bf$$

and

$$BFf = \varphi^{-1}(FFf) = \varphi^{-1}(Ff) = Bf.$$ 

So $EB = B = BF$. Similarly, $FA = A = AE$.

Now define

$$T = \begin{bmatrix} A & I_n - F \\ I_n - E & B \end{bmatrix} \in M_{2n}(C(X)).$$

With the observations above it is straightforward to check that $T$ is invertible, with inverse

$$T^{-1} = \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix}.$$

Then

$$T \text{diag}(E, 0_n)T^{-1} = \begin{bmatrix} A & I_n - F \\ I_n - E & B \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B & I_n - E \\ I_n - F & A \end{bmatrix}$$

$$= \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} = \text{diag}(F, 0_n)$$

Thus $E$ is similar to $F$ through an invertible matrix $T$. Since $E$ and $F$ are normal and similar to each other, they are in fact unitarily equivalent by Proposition 3.14.

**Corollary 8.5.** Let $X$ be compact Hausdorff. The range map

$$\text{Ran} : \mathcal{P}_\infty(C(X))/\sim_u \rightarrow \text{Vect}(X)$$

39
mapping
\[ [E] \mapsto [\text{Ran } E] \]
is well-defined and injective.

**Proposition 8.6.** Let \( X \) be a compact Hausdorff space, let \( N \in \mathbb{N} \), and suppose that \( V \) is a subbundle of \( \Theta^N(X) \). Let \( \Theta^N(X) \) be equipped with the standard Hermitian metric, and for \( x \in X \), let \( E(x) \) be the orthogonal projection of \( \Theta^N(X)\vert_x \) onto \( V\vert_x \). Then the map \( E : x \mapsto E(x) \) defines an idempotent \( E \in M_N(C(X)) \).

**Proof.** By using Lemma 7.7 again, we only need to show that each \( x_0 \in X \) has an open neighbourhood for which \( E \vert_U : x \mapsto E(x) \) is continuous on \( U \).

Fix \( x_0 \) and choose \( U \) to be a connected open neighbourhood of \( x_0 \) over which \( V \) is trivial. Let \( n \) be the rank of \( V \), and let \( \varphi : \Theta^n(U) \rightarrow V\vert_U \) be a bundle isomorphism. For \( k = 1, \ldots, n \), define \( s_k : U \rightarrow \Theta^n(U) \) by \( s_k(x) = (x, e_k) \), the \( k^{th} \) standard basis vector lying above \( x \). Then for each \( x \in U \), the set
\[
\{ \varphi(s_1(x)), \varphi(s_2(x)), \ldots, \varphi(s_n(x)) \}
\]
is a vector space basis for \( V\vert_x \). Let \( \langle \cdot, \cdot \rangle \) be the standard Hermitian metric of \( \Theta^n(U) \) restricted to \( V \). By the Gram-Schmidt process, we obtain a an orthogonal basis of \( V\vert_x \) by defining inductively
\[
s'_k(x) = \varphi(s_k(x)) - \sum_{i=1}^{k-1} \frac{\langle \varphi(s_k(x)), s'_i(x) \rangle}{\langle s'_i(x), s'_i(x) \rangle} s'_i(x)
\]
for \( k = 1, \ldots, n \). Then the set
\[
\left\{ \frac{s'_1(x)}{\|s'_1(x)\|}, \ldots, \frac{s'_n(x)}{\|s'_n(x)\|} \right\}
\]
is an orthonormal basis for \( V\vert_x \) equipped with \( \langle \cdot, \cdot \rangle \), where \( \| \cdot \| \) denotes the norm induced by \( \langle \cdot, \cdot \rangle \). Moreover, the map \( x \mapsto \frac{s'_k(x)}{\|s'_k(x)\|} \) is continuous. Finally, for \( E \) the orthogonal projection as in the statement, we have
\[
E(x)w = \sum_{k=1}^n \left\langle \varphi(x, w), \frac{s'_k(x)}{\|s'_k(x)\|} \right\rangle \frac{s'_k(x)}{\|s'_k(x)\|}
\]
and the above is jointly continuous in \( x \in X \) and \( w \in \mathbb{C}^n \). Therefore \( x \mapsto E(x) \) is continuous. \( \blacksquare \)
Corollary 8.7. Let $V$ be a vector bundle over a compact Hausdorff space $X$. Then $V \cong \text{Ran } E$ for some $E \in \mathcal{P}_\infty(C(X))$. Hence the map

$$\text{Ran} : \mathcal{P}_\infty(C(X))/\sim_u \to \text{Vect}(X)$$

is surjective.

Proof. There exists $N \in \mathbb{N}$ such that $V$ is isomorphic to a subbundle of $\Theta^N(X)$. So assume that $V$ is embedded in $\Theta^N(X)$, and let $\Theta^N(X)$ be equipped with the canonical metric. For each $x \in X$ let $E(x)$ be the orthogonal projection of $\Theta^N(X)_x$ onto $V_x$. By Proposition 8.6, $x \mapsto E(x)$ defines an element in $E \in \mathcal{P}_N(C(X))$. Define $V^\perp = \text{Ran}(I_N - E)$. Then

$$V \oplus V^\perp \cong \text{Ran } E \oplus \text{Ran } (I_N - E) = \text{Ran } I_N = \Theta^N(X).$$

Corollary 8.8. Let $V$ be a vector bundle over a compact Hausdorff space $X$. Then there exists another vector bundle $V^\perp$ over $X$ such that $V \oplus V^\perp \cong \Theta^N(X)$ for some $N \in \mathbb{N}$.

Proof. We know that there exists some $N \in \mathbb{N}$ such that $V$ is isomorphic to a subbundle of $\Theta^N(X)$. For each $x \in X$, let $E(x)$ be the orthogonal projection of $\Theta^N(X)_x$ onto $V_x$. By Proposition 8.6, this family of projections defines an element $E \in \mathcal{P}_N(C(X))$. Define $V^\perp = \text{Ran } (I_N - E)$. Then

$$V \oplus V^\perp \cong \text{Ran } E \oplus \text{Ran } (I_N - E) = \text{Ran } I_N = \Theta^N(X).$$

Theorem 8.9. Let $X$ be a compact Hausdorff space. Then $\mathcal{P}_\infty(C(X))$ and $\text{Vect}(X)$ are isomorphic as abelian monoids.

Proof. Define $\Psi : \mathcal{P}_\infty(C(X)) \to \text{Vect}(X)$ by $\Psi([E]) = [\text{Ran } E]$. By Corollaries 8.5 and 8.7, $\Psi$ is well-defined, injective and surjective. It is left to show that it is a monoid homomorphism, i.e. $\text{Ran } (E \oplus F) \cong \text{Ran } E \oplus \text{Ran } F$. But this is obvious, as they are not just isomorphic, but are in fact equal.

Corollary 8.10. Let $X$ be a compact Hausdorff space. Then $K^0(X) \cong K_0(C(X))$ as abelian groups.

Proof. Apply the Grothendieck completion to the isomorphism obtained in Theorem 8.9 to obtain

$$K^0(X) = G(\text{Vect}(X)) \cong G(\mathcal{P}_\infty(C(X))) = K_0(C(X)).$$
For $X$ a compact Hausdorff space and $V$ a topological vector bundle over $X$, we write $[V]^0$ for the element in $K^0(X)$ that is represented by $V$.

**Proposition 8.11.** Let $X$ be a compact Hausdorff space, then

$$K^0(X) = \{ [V]^0 - [W]^0 : V, W \text{ vector bundles over } X \}.$$  

**Proof.** This follows from Corollary 8.10 and Proposition 4.3.  

Now that we’ve shown that $K^0(X)$ and $K_0(C(X))$ are isomorphic as abelian groups, we will verify that the associated morphisms are preserved by this identification.

**Definition 8.12.** Let $X$ and $Y$ be compact Hausdorff spaces, let $f : X \to Y$ be a continuous map and let $V$ be a rank $r$ subbundle of some trivial bundle $\Theta^n(Y)$ of $Y$. (By Proposition 7.8 all vector bundles over $Y$ are isomorphic to a bundle of this form). Define the pull-back of $V$ via $f$, written $f^*(V)$, to be the rank $r$ subbundle of $\Theta^n(X)$ where the fibre at a point $x \in X$ is $(f^*(V))_x = V_{f(x)}$.

**Proposition 8.13.** Let $X$ and $Y$ be compact Hausdorff spaces, $f : X \to Y$ continuous and $V$ is a subbundle of $\Theta^n(Y)$. Then $f^*(V)$ is indeed a vector bundle on $X$.

**Proof.** Take any $x \in X$, let $U$ be an open neighbourhood of $f(x)$ in $Y$ for which $V|_U$ is trivial. Then $f^{-1}(U)$ is an open neighbourhood of $x$ and $f^*(V)|_{f^{-1}(U)} = f^*(V|_U)$ is trivial.  

**Proposition 8.14.** Let $X$ and $Y$ be compact Hausdorff spaces, let $f : X \to Y$ be continuous, and $E \in \mathcal{P}_\infty(C(Y))$. Then $f^*(E)$ is a projection in $\mathcal{P}_\infty(C(X))$, and $f^*(\text{Ran } E) = \text{Ran } f^*(E)$.

**Proof.** For $x \in X$,

$$(E \circ f)(x) \cdot (E \circ f)(x) = E(f(x))E(f(x)) = EE(f(x)) = E \circ f(x)$$

and

$$(E \circ f)^*(x) = (E \circ f(x))^* = E^*(f(x)) = E \circ f(x).$$
So \( E \circ f \) is a projection. Furthermore, suppose \( E \) is \( n \times n \). Then
\[
f^*(\text{Ran } E)_x = (\text{Ran } E)_{f(x)} = E(f(x))C^n = (\text{Ran } f^*(E))_x.
\]
Therefore \( f^*(\text{Ran } E) = \text{Ran } f^*(E) \).

**Definition 8.15.** Let \( X \) and \( Y \) be compact Hausdorff spaces and let \( f : X \to Y \) be a continuous map. Then \( f^* \) is a *-homomorphism from \( C(Y) \) to \( C(X) \). Define \( K^0(f) : K^0(Y) \to K^0(X) \) by
\[
K^0(f)([V]_0 - [W]_0) = [f^*(V)]_0 - [f^*(W)]_0.
\]

**Remark 8.16.** According to Proposition 8.14, if \( f : X \to Y \) is a continuous map then by identifying \( K^0(Y) \) with \( K^0(C(Y)) \) and \( K^0(X) \) with \( K^0(C(X)) \), we conclude that \( K^0(f) \) and \( K^0(f^*) \) are the same map. To be precise, the diagram
\[
\begin{array}{ccc}
K^0(Y) & \xrightarrow{K^0(f)} & K^0(X) \\
\downarrow \cong & & \downarrow \cong \\
K_0(C(Y)) & \leftarrow & K_0(C(X)) \end{array}
\]
commutes.

**Proposition 8.17.** The map \( X \mapsto K^0(X) \) is a covariant functor from the category of compact Hausdorff spaces to the category of abelian groups.

**Proof.** Let \( X, Y, Z \) be compact Hausdorff spaces, and let \( f : X \to Y \) and \( g : Y \to Z \) be continuous. Consider the commutative diagrams
\[
\begin{array}{ccc}
K^0(Z) & \xrightarrow{K^0(g)} & K^0(Y) & \xrightarrow{K^0(f)} & K^0(X) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
K_0(C(Z)) & \leftarrow & K_0(C(Y)) & \leftarrow & K_0(C(X)) \end{array}
\]

and
\[
\begin{array}{ccc}
K^0(Z) & \xrightarrow{K^0(f \circ g)} & K^0(X) \\
\downarrow \cong & & \downarrow \cong \\
K_0(C(Z)) & \leftarrow & K_0(C(X)) \end{array}
\]
Since $K_0$ is a functor, we have

$$K_0((f \circ g)^*) = K_0(g^* \circ f^*) = K_0(g^*) \circ K_0(f^*).$$

Hence the first rows of the two diagrams imply that $K^0(f \circ g) = K^0(g) \circ K^0(f)$. The fact that $K^0(\text{id}_X) = \text{id}_{K^0(X)}$ also follows from the functoriality of $K_0$ and Remark 8.16 in a similar way. 

**Example 8.18.** Let $X = \{\ast\}$ be a point. Then $C(X) \cong \mathbb{C}$. By Example 2.18 and Corollary 8.10, we see that $K^0(X) \cong K^0(\mathbb{C}) \cong \mathbb{Z}$.

### 9 K-theory of locally compact spaces

The K-theory of locally compact spaces correspond to the K-theory of non-unital C*-algebras.

**Definition 9.1.** Let $X$ be a topological space. We say $X$ is locally compact if for every $x \in X$ there exists some open neighbourhood $U \subseteq X$ of $x$ such that the closure $\overline{U}$ of $U$ in $X$ is compact.

**Definition 9.2.** Let $X$ be a locally compact space. Define $X^+$ to be the set $X \sqcup \{\infty\}$ with the collection of open sets given by

$$\mathcal{T}^+ := \{U \subseteq X : U \text{ open in } X\} \cup \{(X \setminus F) \cup \{\infty\} : F \text{ closed and compact in } X\}.$$

**Proposition 9.3.** Let $X$ be a topological space, then $X^+$ is a compact topological space. Moreover, $X^+ \setminus \{\infty\}$ is homeomorphic to $X$ in the obvious way.

**Proof.** We first check that the collection of open sets $\mathcal{T}^+$ is a topology on $X^+$.

1. The empty set $\emptyset$ is open in $X$, so $\emptyset \in \mathcal{T}^+$. The empty set $\emptyset$ is obviously closed and compact, so $X^+ = (X \setminus \emptyset) \cup \{\infty\} \in \mathcal{T}^+$.

2. Define

$$\mathcal{T}_0 := \{U : U \text{ open in } X\},$$

$$\mathcal{T}_1 := \{(X \setminus F) \cup \{\infty\} : F \text{ closed and compact in } X\}.$$  

Clearly $\mathcal{T}_0$ is closed under arbitrary union. Let $\{F_i : i \in I\}$ be an arbitrary collection of closed compact subsets of $X$. Then $F := \bigcap_{i \in I} F_i$ is clearly
closed. Pick any \( i_0 \in I \). Then \( F \) is a closed subset of the compact set \( F_{i_0} \), thus \( F \) is also compact. Then

\[
\bigcup_{i \in I} (X \setminus F_i) \cup \{\infty\} = (X \setminus F) \cup \{\infty\} \in \mathcal{T}_1.
\]

So \( \mathcal{T}_1 \) is closed under arbitrary union. Finally, take \( U \in \mathcal{T}_0 \) and \( (X \setminus F) \cup \{\infty\} \in \mathcal{T}_1 \). We have

\[
U \cup (X \setminus F) \cup \{\infty\} = (X \setminus (X \setminus U)) \cup (X \setminus F) \cup \{\infty\} = (X \setminus ((X \setminus U) \cap F)) \cup \{\infty\} \in \mathcal{T}_1
\]

because \( (X \setminus U) \cap F \) is closed and compact (it is a closed subset of \( F \)). Therefore \( \mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \) is closed under arbitrary union.

3. Clearly \( \mathcal{T}_0 \) is closed under finite intersection. A finite union of compact closed sets is also closed and compact, so \( \mathcal{T}_1 \) is also closed under finite intersection. Lastly, suppose \( U \) is open and \( F \) is closed and compact, then

\[
U \cap ((X \setminus F) \cup \{\infty\}) = U \cap (X \setminus F) \in \mathcal{T}_1.
\]

Therefore \( \mathcal{T} \) is closed under finite intersection.

The above verifies that \( \mathcal{T} \) is a topology on \( X \). The subspace topology on \( X^+ \setminus \{\infty\} \) is \( \mathcal{T}_0 \), which coincides with the topology on \( X \). Hence \( X^+ \setminus \{\infty\} \cong X \). Next we check that \( X^+ \) is compact.

Let \( \{U_i\}_{i \in I} \) be a open cover for \( X^+ \). Since this collection covers the point \( \infty \), there exists some \( i_0 \in I \) such that \( U_{i_0} \in \mathcal{T}_1 \). Then \( X^+ \setminus U_{i_0} \) is a compact subset of \( X \), hence also a compact subset of \( X^+ \), so there exists a finite subset \( J \subseteq I \) for which \( X^+ \setminus U_{i_0} \subseteq \bigcup_{i \in J} U_i \). Whence \( \{U_i : i \in J \cup \{i_0\}\} \) is a finite cover for \( X^+ \). Therefore \( X^+ \) is compact.

Remark 9.4. The space \( X^+ \) is called the one point compactification of \( X \).

Proposition 9.5. Let \( X \) be a locally compact topological space. If \( X \) is Hausdorff then \( X^+ \) is also Hausdorff.

Proof. Let \( \mathcal{T}_0 \) be \( \mathcal{T}_1 \) be as defined in the proof of Proposition 9.3. By Proposition 9.3 we know that \( X^+ \setminus \{\infty\} \cong X \) is Hausdorff. Fix \( x \in X^+ \setminus \{\infty\} \) and let \( U \) be an open neighbourhood of \( x \) where \( \overline{U} \) is compact in \( X \). Then \( V := X^+ \setminus \overline{U} \) is an open neighbourhood of \( \infty \), and \( U \cap V = \emptyset \). Therefore \( X^+ \) is Hausdorff.
Proposition 9.6. Let $X$ be a compact Hausdorff space, and let $x_0 \in X$. The map $f : X \to (X \setminus x_0)^+$ given by

$$ f(x) = \begin{cases} x & : x \neq x_0 \\ \infty & : x = x_0 \end{cases} $$

is a homeomorphism.

Proof. It is clear that $f$ is bijective. It is also clear that for any $S \subseteq X \setminus \{x_0\}$, $S$ is open in $X$ if and only if $f(S)$ is open in $(X \setminus \{x_0\})^+$.

Suppose $U \subseteq X$ is an open neighbourhood of $x_0$. Let $F = X \setminus U$. Since $F$ is a closed subset of $X$, it is compact. Also,

$$ U = ((X \setminus \{x_0\}) \setminus F) \cup \{x_0\}. $$

On the other hand, suppose $F \subseteq X \setminus \{x_0\}$ is closed and compact, then

$$ ((X \setminus \{x_0\}) \setminus F) \cup \{\infty\} = X \setminus F $$

is an open neighbourhood of $x_0$. Hence $x_0 \in X$ and $\infty \in (X \setminus \{x_0\})^+$ have the “same” open neighbourhoods. It follows that a subset $S \subseteq X$ containing $x_0$ is open if and only if $f(S)$ is open. Therefore $f$ is a homeomorphism. $\blacksquare$

Definition 9.7. Let $X$ be a locally compact Hausdorff space. Define $C_0(X)$ to be the set of all continuous functions $f \in C(X)$ satisfying the following: for any $\varepsilon > 0$ there exists a compact subset $F \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus F$.

Proposition 9.8. Let $X$ be a locally compact Hausdorff space and let $f \in C_0(X)$. Define $\tilde{f}$ on $X^+$ to be

$$ \tilde{f} = \begin{cases} f(x) & : x \in X \\ 0 & : x = \infty \end{cases}. $$

Then $\tilde{f} \in C(X^+)$. If $h \in C(X^+)$ satisfies $h(\infty) = 0$, then $h|_X \in C_0(X)$ and $\tilde{h}|_X = h$. 

46
Proof. It is clear that $\widetilde{f}$ is continuous on $X^+ \setminus \{\infty\}$, so we only need to check that $\widetilde{f}$ is continuous at $\infty$. Given any $\varepsilon > 0$, by the definition of $C_0(X)$, there exists a compact subset $F \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus F$. But $U := (X \setminus F) \cup \{\infty\}$ is an open neighbourhood of $\infty$. We have $|\widetilde{f}(x) - \widetilde{f}(\infty)| = |\widetilde{f}(x)| < \varepsilon$ for all $x \in U$. Therefore $\widetilde{f}$ is continuous.

The second part of the proof follows essentially the same proof.

**Proposition 9.9.** Let $X$ be a locally compact Hausdorff space. Let $I_X$ denote the identity element of $\widetilde{C}_0(X)$ and let $1_{X^+}$ denote the constant function $1$ on $X^+$. Define $\varphi : \widetilde{C}_0(X) \to C(X^+)$ by $\varphi(f) = \widetilde{f}$ for all $f \in C_0(X)$ and $\varphi(I) = \varphi(1_{X^+})$ and extend linearly. Then $\varphi$ is a $C^*$-algebra isomorphism.

**Proof.** It is easy to see that $\varphi$ is a $\ast$-homomorphism. Suppose

$$0 = \varphi(f + zI_X) = \widetilde{f} + z1_{X^+}$$

for some $f \in C_0(X)$ and $z \in \mathbb{C}$. Then

$$z = (\widetilde{f} + z1_{X^+})(\infty) = 0.$$ 

It then follows that $\widetilde{f}(x) = 0$ for all $x \in X$, so $f = 0$. Hence $\varphi$ is injective.

Take any $h \in C(X^+)$ and let $z = h(\infty)$. By Proposition 9.8 the function $(h - z1_{X^+})|_X \in C_0(X)$. Also, $\varphi((h - z1_{X^+}) + zI_X) = h$. This shows that $\varphi$ is surjective. Therefore $\varphi$ is an isomorphism.

**Definition 9.10.** Let $X$ be a locally compact Hausdorff space, and let $\iota : \{\infty\} \to X^+$ be the inclusion map. Define $K^0(X) := \ker K^0(\iota) \subseteq K^0(X^+)$. 

**Remark 9.11.** Suppose $X$ is a locally compact Hausdorff space and $\iota : \{\infty\} \to X^+$ is the inclusion map. The induced $\ast$-homomorphism $\iota^* : C(X^+) \to C(\{\infty\})$ does the following:

$$\iota^*(\widetilde{f}) = \widetilde{f} \circ \iota = 0, \ \forall f \in C_0(X)$$

and

$$\iota^*(1_{X^+}) = 1_{X^+} \circ \iota = 1_{\{\infty\}}.$$ 

This means that $\iota : C(X^+) \to C(\{\infty\})$ is the projection onto the one dimensional subspace generated by the identity element and $\ker \iota = C_0(X)$. Whence in light of Remark 8.16 and Proposition 9.9, $K^0(X)$ is isomorphic to $K_0(C_0(X))$ in the expected way.
9.1 Relative and reduced K-theory

**Definition 9.12.** Let $X$ be a compact Hausdorff space, and let $A$ be a compact subset of $X$. Let $\iota: A \to X$ be the inclusion map. Then $K^0(\iota)$ is a group homomorphism $K^0(X) \to K^0(A)$. Define $K^0(X, A)$ to be $\ker(K^0(\iota))$. The group $K^0(X, A)$ is called the relative K-group of the compact pair $(X, A)$.

**Proposition 9.13.** Let $X$ be a locally compact Hausdorff space. Then $K^0(X) \cong K^0(X^+, \infty)$.

*Proof.* This is a consequence of Remark 9.11. □

**Proposition 9.14.** Let $X$ be a compact Hausdorff space and fix $x_0 \in X$. Then $K^0(X) \cong K^0(X, x_0) \oplus \mathbb{Z}$.

*Proof.* Let $\iota: \{x_0\} \to X$ be the inclusion map, and let $\lambda: X \to \{x_0\}$ be the only constant map. Consider the sequence

$$
0 \longrightarrow K^0(X, x_0) \longrightarrow K^0(X) \xrightarrow{K^0(\iota)} K^0(\{x_0\}) \longrightarrow 0.
$$

By the definition of $K^0(X, x_0)$, this sequence is exact. Furthermore, $\iota \circ \lambda = \text{id}_{\{x_0\}}$, then by the functoriality of $K^0$ we have that

$$K^0(\lambda) \circ K^0(\iota) = K^0(\iota \circ \lambda) = K^0(\text{id}_{\{x_0\}}) = \text{id}_{K^0(\{x_0\})}.$$

Hence the above is a split exact sequence of abelian groups. Therefore $K^0(X) \cong K^0(X, x_0) \oplus K^0(\{x_0\})$. Lastly, by Example 8.18 we have $K^0(\{x_0\}) \cong \mathbb{Z}$. □

**Remark 9.15.** Let $X$ be a compact Hausdorff space. Let $G_0$ be the subgroup of $K^0(X)$ generated by $[\Theta^1(X)]_0$. Since

$$[\Theta^n(X)]_0 + [\Theta^m(X)]_0 = [\Theta^n(X) \oplus \Theta^m(X)]_0 = [\Theta^{n+m}(X)]_0,$$

we have that $G_0 = \{\pm[\Theta^n(X)]_0 : n \in \mathbb{N}_{\geq 0}\} \cong \mathbb{Z}$. Fix $x_0 \in X$, and let $\iota_{x_0}: \{x_0\} \to X$ be the inclusion map. Then

$$K^0(\iota_{x_0})([\Theta^n(X)]_0) = [\iota_{x_0}^*([\Theta^n(X)]_0)]_0 = [\Theta^n(\{x_0\})]_0,$$
which corresponds to \( n \in \mathbb{Z} \) in the isomorphism \( \mathbb{Z} \cong K^0(\{x_0\}) \). Hence \( K^0(t_{x_0})|_{G_0} \to K^0(\{x_0\}) \) is an isomorphism for any \( x_0 \in X \). Thus we have that \( K^0(X, x_0) \cong K^0(X)/G_0 \) for any \( x_0 \in X \). More importantly, we have that \( K^0(X, x_0) \cong K^0(X, x_1) \) for any \( x_0, x_1 \in X \).

**Definition 9.16.** Let \( X \) be a compact Hausdorff space. Define the reduced \( K \)-group \( \tilde{K}^0(X) \), to be \( K^0(X, x_0) \) for any choice of \( x_0 \in X \).

**Remark 9.17.** Let \( X \) be a compact Hausdorff space and fix \( x_0 \in X \). By Proposition 9.13 we have \( \tilde{K}^0(X) \cong K^0(X, \{x_0\}) \). By Remark 9.15, the definition of \( \tilde{K}^0(X) \) is independent of the choice \( x_0 \in X \).

## 10 Functorial properties of \( K^0 \)

### 10.1 Homotopy invariance

**Definition 10.1.** Let \( X \) and \( Y \) be topological spaces and let \( f, g : X \to Y \) be continuous maps. We say \( f \) is homotopic to \( g \) if there exists a continuous map \( f_t : [0, 1] \times X \to Y \) mapping \( (t, x) \mapsto f_t(x) \) such that \( f_0(x) = f(x) \) and \( f_1(x) = g(x) \) for all \( x \in X \).

**Definition 10.2.** Let \( X \) and \( Y \) be topological spaces. Then \( X \) is said to be homotopic to \( Y \) if there exist continuous maps \( f : X \to Y \) and \( g : Y \to X \) such that \( f \circ g \) is homotopic to id\(_Y \) and \( g \circ f \) is homotopic to id\(_X \).

**Lemma 10.3.** Let \( X \) and \( Y \) be compact Hausdorff spaces, and let \( \varphi : [0, 1] \times X \to Y \) mapping \( (t, x) \mapsto \varphi_t(x) \) be continuous. Then the map \( t \mapsto (\varphi_t)^*(f) = f \circ \varphi_t \) is continuous from \( [0, 1] \) to \( C(X) \) for any \( f \in C(Y) \).

**Proof.** Let \( f \in C(Y) \) and \( \varepsilon > 0 \) be given. Then \( f \circ \varphi : [0, 1] \times X \to \mathbb{R} \) is a continuous function. By continuity, for any \( t \in [0, 1] \) and \( x \in X \), there exists \( \delta_t > 0 \) and an open neighbourhood \( U_x \subseteq X \) of \( x \) such that

\[
|f \circ \varphi_s(y) - f \circ \varphi_t(x)| < \varepsilon
\]

for every \( s \in B_{\delta_t}(t) \cap [0, 1] \) and \( y \in U_x \). By compactness, \( X \) can be covered by a finite collection of open sets of the form \( U_{x_1}, \ldots, U_{x_k} \). Let \( \delta = \min\{\delta_{t_1}, \ldots, \delta_{t_k}\} > 0 \). Then for any \( x \in X \),

\[
|(\varphi_s)^*(f)(x) - (\varphi_t)^*(f)(x)| = |f \circ \varphi_s(x) - f \circ \varphi_t(x)| < \varepsilon,
\]

so \( \|(\varphi_s)^*(f) - (\varphi_t)^*(f)\|_\infty < \varepsilon \).  

---

49
Proposition 10.4. Let $X$ and $Y$ be compact Hausdorff spaces. Let $f : X \to Y$ and $g : Y \to X$ be a homotopy between $X$ and $Y$. Then $f^* : C(Y) \to C(X)$ and $g^* : C(X) \to C(Y)$ give a homotopy between $C(X)$ and $C(Y)$.

Proof. By assumption $g \circ f$ is homotopic to the identity map $\text{id}_X$ on $X$. Hence there exists a continuous family $\varphi_t : X \to X$ for $t \in [0, 1]$ satisfying $\varphi_0 = \text{id}_X$ and $\varphi_1 = g \circ f$. By Lemma 10.3, $(\varphi_\bullet)^*$ is a homotopy from $(\varphi_0)^* = (\text{id}_X)^* = \text{id}_{C(X)}$ to $(\varphi_1)^* = (g \circ f)^* = f^* \circ g^*$. Similarly $g^* \circ f^*$ is homotopic to $\text{id}_{C(Y)}$.

Corollary 10.5. Let $X$ and $Y$ be compact Hausdorff spaces and $f : X \to Y$ be a homotopy. Then $K^0(f) : K^0(Y) \to K^0(X)$ is a group isomorphism.

Proof. By Proposition 10.4 we see that $f^* : C(Y) \to C(X)$ is a homotopy. It follows by Proposition 6.2 that $K^0(f^*)$ is an isomorphism, whence Remark 8.16 gives us the conclusion that $K^0(f)$ is an isomorphism.

Example 10.6. Let $X = [0, 1]$. Then $X$ is homotopic to a point. Hence by Corollary 10.5 and Example 8.18, we have

$$K_0(C([0, 1])) \cong K^0([0, 1]) \cong K^0(\{\ast\}) \cong \mathbb{Z}.$$  

Remark 10.7. The functor $K^0$ is not homotopy-invariant for locally compact Hausdorff spaces. In Example 7.17 we saw that $K^0(S^1) \cong \mathbb{Z}$. The unit circle $S^1$ is homeomorphic to the one point compactification of $\mathbb{R}$, and $\mathbb{R}$ is homotopic to a point. However, Proposition 9.14 says that $K^0(\mathbb{R}) \oplus \mathbb{Z} \cong K^0(S^1)$, which implies that $K^0(\mathbb{R}) \cong 0$. On the other hand, the $K^0$-group of a point is $\mathbb{Z}$, as shown in Example 10.6, which is not isomorphic to $K^0(\mathbb{R})$.

Example 10.8. We will now exhibit an example that shows $K_0$ is not an exact functor.

Consider the short exact sequence

$$0 \longrightarrow C_0((0, 1)) \xrightarrow{\iota} C([0, 1]) \xrightarrow{\pi} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

Where

$$(\iota(f))(t) := \begin{cases} f(t) & : t \in (0, 1) \\ 0 & : t \in \{0, 1\} \end{cases}$$

50
for any $f \in C_0((0,1))$ and $t \in [0,1]$, and
\[
\pi(g) := (g(0), g(1))
\]
for any $g \in C([0,1])$. It is left to the reader to check that this sequence is exact.

Corollary 6.7 and Example 8.18 give us the isomorphism
\[
K_0(C \oplus C) \cong K_0(C) \oplus K_0(C) \cong \mathbb{Z} \oplus \mathbb{Z}.
\]
On the other hand $C([0,1]) \cong \mathbb{Z}$ by Example 10.6. The map $K_0(\pi) : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is not a surjection, since $\mathbb{Z}$ is generated by one element but $\mathbb{Z} \oplus \mathbb{Z}$ cannot be generated by one element. Therefore the functor $K_0$ does not take the short exact sequence in consideration to a short exact sequence of abelian groups.

10.2 Half-exactness of $\tilde{K}^0$

**Proposition 10.9.** Let $X$ be a compact Hausdorff space and let $A$ be a closed subset of $X$. Define $I(A)$ to be all the continuous functions $f \in C(X)$ that vanish on $A$, i.e. $f(A) = \{0\}$. Then the following are true

1. $I(A)$ is a closed ideal of $C(X)$.
2. $I(A) \cong C_0(X \setminus A)$.
3. Let $[A]$ denote the point corresponding to $A$ in the quotient $X/A$. Then $(X/A) \setminus \{[A]\} \cong X \setminus A$ as locally compact Hausdorff spaces.
4. $I(A) \cong C_0((X/A) \setminus \{[A]\})$.
5. $C(X)/I(A) \cong C(A)$.

**Proof.**
1. Let $f \in I(A)$ and $g \in C(X)$, then
\[
(f \cdot g)(a) = f(a)g(a) = 0g(a) = 0
\]
for all $a \in A$, so $f \cdot g \in I(A)$. Clearly if a convergent sequence of functions vanish on $A$ then so does the limit. Hence $I(A)$ is a closed ideal in $C(X)$. 

51
2. Let \( \varphi : C_0(X \setminus A) \to C(X) \) be defined by

\[
\varphi(f)(x) = \begin{cases} 
  f(x) & : x \in X \setminus A \\
  0 & : x \in A
\end{cases}
\]

for all \( f \in C_0(X \setminus A) \) and \( x \in X \). For each \( \varepsilon > 0 \), there exists an open neighbourhood \( U \subseteq X \) with \( A \subseteq U \) satisfying \(|\varphi(f)(x)| < \varepsilon\) for all \( x \in U \). Hence we see that \( \varphi(f) \in C(X) \) for all \( f \in C_0(X \setminus A) \). It is also clear from definition that the image of \( \varphi \) is contained in \( I(A) \). We also define a map \( \psi : I(A) \to C(X \setminus A) \) by

\[
\psi(g)(x) = g(x)
\]

for all \( g \in I(A) \) and \( x \in X \setminus A \). Since \( g(A) = \{0\} \), then for every \( \varepsilon > 0 \) there exists an open neighbourhood \( U \supseteq A \) satisfying \(|g(x)| < \varepsilon\) for all \( x \in U \). Hence \( \psi(g) \in C_0(X \setminus A) \). It is easy to check that \( \varphi \) and \( \psi \) are mutual inverses. Therefore

\[
C_0(X \setminus A) \cong I(A).
\]

3. This is obvious.

4. This is a consequence of 2 and 3.

5. Define \( \varphi : C(X)/I(A) \to C(A) \) by letting \( \varphi([f]) = f|_A \). If \( [f] = [g] \), then \( (f - g)|_A = 0 \), so \( \varphi([f]) = \varphi([g]) \). Hence \( \varphi \) is well-defined.

Define \( \psi : C(A) \to C(X)/I(A) \) as follows. Fix \( h \in C(A) \), by Tietze’s extension theorem [7] the function \( \tilde{h} \) extends to a continuous function \( \tilde{h} \in C(X) \). Let \( \psi(h) = [\tilde{h}] \). It is easy to check that \( \varphi \) and \( \psi \) are mutual inverses. Therefore

\[
C(A) \cong C(X)/I(A).
\]

**Corollary 10.10.** Let \( X \) be a compact Hausdorff space and let \( A \) be a closed subset of \( X \). Under the identifications \( I(A) \cong C_0(X \setminus A) \) and \( C(A) \cong C(X)/I(A) \), the following sequence is exact:

\[
K_0(C_0((X/A) \setminus \{[A]\})) \longrightarrow K_0(C(X)) \longrightarrow K_0(C(A))
\]

**Proof.** Consider the following diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & I(A) & \longrightarrow & C(X) & \longrightarrow & C(X)/I(A) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow = & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
0 & \longrightarrow & C_0((X/A) \setminus \{[A]\}) & \longrightarrow & C(X) & \longrightarrow & C(A) & \longrightarrow & 0
\end{array}
\]

52
The upper row is clearly exact. The isomorphisms from the upper row to the lower row are given by Lemma 10.9. By the half-exactness of the functor $K_0$ 6.5, we obtain the exactness of the $K_0$-groups. \[\square\]

**Corollary 10.11.** Let $X$ be a compact Hausdorff space and let $A$ be a closed subset of $X$. Let $\iota : A \to X$ be the inclusion map and let $\pi : X \to X/A$ be the projection map. The following sequence is exact:

$$
\tilde{K}^0(X/A) \xrightarrow{K^0(\pi)} K^0(X) \xrightarrow{K^0(\iota)} K^0(A).
$$

*Proof.* By Corollary 8.10, we know $K^0(X) \cong K_0(C(X))$ and $K^0(A) \cong K_0(C(A))$. By Remark 9.17 and Remark 9.11, we have that $K_0((X/A) \setminus \{[A]\}) \cong K^0((X/A) \setminus \{[A]\}) \cong \tilde{K}^0(X/A)$. To see that $K^0(\pi)$ and $K^0(\iota)$ are the maps in this exact sequence, one can take $\pi$ and $\iota$ and chase through the proofs in this section. \[\square\]

**Remark 10.12.** The functor $K^0$ is not half-exact. If $A$ is a compact subset of a compact Hausdorff space $X$ and we take the quotient $X/A$, the subspace $A$ is contracted to a point rather than deleted, and this point is not present in the corresponding C*-algebra quotient. The point in $X/A$ representing $A$ detects the rank of the bundles, so we take the reduced $\tilde{K}^0$ to delete this extra information and make the sequence exact.

**Proposition 10.13.** Let $X$ and $Y$ be locally compact Hausdorff spaces. Then

$$K^0(X) \oplus K^0(Y) \cong K^0(X \sqcup Y).$$

*Proof.* It can be easily verified that $C(X) \oplus C(Y) \cong C(X \sqcup Y)$. By Corollaries 6.7 and 8.10 we have

$$K^0(X) \oplus K^0(Y) \cong K_0(C(X)) \oplus K_0(C(Y)) \cong K_0(C(X) \oplus C(Y)) \cong K^0(X \sqcup Y).$$

11 What’s next

Computing the $K_0$ or $K^0$ group can be very difficult even with the machinery we have developed. The next step is to define the higher $K$-groups by

$$K_{n+1}(A) := K_n(SA) \text{ or } K^{n-1}(X) := K^n(SX),$$

where $S$ denotes the suspension.
of the C*-algebra or the topological space. The isomorphism $K_n(C(X)) \cong K^{-n}(X)$ holds for all $n$. For a C*-algebra and a closed ideal $I$, there exist connecting maps for which the long sequence

$$\ldots \rightarrow K_2(A/I) \rightarrow K_1(I) \rightarrow K_1(A) \rightarrow K_1(A/I) \rightarrow K_0(I) \rightarrow K_0(A) \rightarrow K_0(A/I)$$

is exact. The corresponding sequence is exact for the reduced topological $K$-theory, with arrows pointed in the opposite direction.

The celebrated Bott Periodicity theorem says that $K_n(A) \cong K_{n+2}(A)$ (or $K_n(X) \cong K^{n+2}(X)$) for all $n$. This reduces the above sequence to a sequence with six elements. It also implies that if we know the $K_0$- and $K_1$-group of a C*-algebra then we can read off the $K$-groups of its suspensions. For example, to find the $K$-groups of spheres of any dimension, one only needs to compute $K^0$ and $K^1$ for the two pointed space $S^0$. The interested readers are referred to [1] and [4] for more details.
References


[2] K. R. Davidson, *C*-Algebras by Example, American Mathematical So-
ciety, United States (1996).


