Positive Bisectional Curvature on Compact Kähler surfaces and Kähler–Einstein Manifolds

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Abstract

This research paper investigates holomorphic bisectional curvature and the Frankel conjecture. The Frankel conjecture states that a connected compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to the complex projective space. Following Goldberg and Kobayashi, we present proofs of the conjecture in dimension two and in the case of Kähler–Einstein manifolds.

Contents

Introduction

The Frankel conjecture was posed by Frankel in 1961 and states that a closed Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbb{CP}^n$. Its algebro-geometric version, known as the Hartshorne conjecture, was posed by Hartshorne in 1970 and states that an irreducible non-singular projective variety over an algebraically closed field k, with ample tangent bundle, is isomorphic to a projective space over k. It was proved by Mori using algebraic geometry of charateristic $p > 0$. Hartshorne's conjecture is more general than Frankel's conjecture, as it only requires the tangent bundle to be ample, where as Frankel's conjecture requires positive bisectional curvature. Frankel's conjecture was proved in the two-dimensional case by Andreotti-Frankel, and resolved completely by Siu-Yau using harmonic maps and characterization of projective space obtained by Kobayashi-Ochiai.

A full proof of Frankel's conjecture can be found in Siu and Yau [9]. This research paper will focus on the cases of dimension two and Kähler–Einstein manifolds. The following is a breakdown of the contents.

The first section will be a summary of basic notions. There two sections on manifolds with a slight twist in presentation, and one big section on Kähler structures. All three sections have proofs of most statements. Also, there is one section on Riemannian geometry containing no proofs. References will be given almost throughout when proofs are missing.

The second section will be an exposition of Goldberg and Kobayashi [10]. The Frankel conjecture in dimension two will be proved using complex algebraic geometry and Castelnuovo-Andreotti's result on surface classification. The Kähler-Einstein case will be proved using mainly Riemannian geometry. In the Kähler-Einstein case, a stronger result than Frankel's conjecture can be attained, giving a holomorphic isometry instead of a biholomorphic equivalence.

1 Basic Kähler geometry

1.1 Differentiable and holomorphic maps

Let $\mathbb{K}$ be a field of characteristic 0. For our purposes, we can assume $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The standard coordinates $x^1, \ldots, x^n$ of the vector space $\mathbb{K}^n$ will be realized as the dual of the standard basis $e_1, \ldots, e_n$ of $\mathbb{K}^n$. They will often be used to indicate components of elements in $\mathbb{K}^n$, as $x = x^i(x)e_i$ for every $x \in \mathbb{K}^n$. The Einstein summation convention will be used.

Let $x^1, \ldots, x^n$ be the standard coordinates of $\mathbb{R}^n$.

Definition 1.1.1. Given a continuous map $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ and $a \in U$, the *partial derivative* $\frac{\partial f}{\partial x^i}(a)$ at a is the limit

$$\lim_{t\to 0} \frac{f(x^1, \ldots, x^i + t, \ldots, x^n)}{t},$$

where the limit is taken over the set of $a + te_i \in U$ with $t \in \mathbb{R}$. If $\frac{\partial f}{\partial x^i}(a)$ exists for all $a \in U$, we write $\frac{\partial f}{\partial x^i}$ for the resulting real function on U.

Let $y^1, \ldots, y^m$ be the standard coordinates of $\mathbb{R}^m$. Consider a map $f : U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, denote $f^i = y^i \circ f$.

Definition 1.1.2. The *total derivative* f_* is the matrix given by $(f_*)^i_j = \frac{\partial f^i}{\partial x^j}$.

Remark 1.1.3. Note that if the partial derivatives exist, f_* exists and is a function $U \to M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$. In particular, we can talk about the total derivative of f_* as a function $U \to \mathbb{R}^{n^2}$ which may not be continuous.

Definition 1.1.4. A map $f : U \to V$ is *r-times continuously differentiable* or C^r, if the partial derivatives $\frac{\partial^r f^j}{\partial x^{i_1} \ldots \partial x^{i_r}}$ exists and are continuous, or equivalently, the result of applying the operation of taking the total derivative r-times to f exists and is continuous, and f is *infinitely differentiable* or C^∞ if it is C^r for each $r \in \mathbb{N}$.

Remark 1.1.5. We include 0 in $\mathbb{N}$, and denote $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. By convention f is C^0 if f is continuous.

In the rest of this subsection, we assume the following setup in each definition and proposition. Let $z^1, \ldots, z^n$ be the standard coordinates on $\mathbb{C}^n$. We may write $z^l = x^l + y^l i$, where x^l, y^l are two copies of the coordinates for $\mathbb{R}^n$, and identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by identifying the coordinates via $(z_1, \ldots, z_n) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$, which preserves the topology on the two spaces. Put

$$j^r = \begin{bmatrix} 0 & -I_r \\ I_r & 0 \end{bmatrix} .$$

Fix a map $f : U \to V$, where $U \subseteq \mathbb{C}^m = \mathbb{R}^{2m}$ and $V \subseteq \mathbb{C}^n = \mathbb{R}^{2n}$ are open subsets, and define f_* using this identification between $\mathbb{C}^n$ and $\mathbb{R}^n$. Denote $f^l = z^l \circ f$, $u^l = x^l \circ f$ and $v^l = y^l \circ f$, so $f^l = u^l + v^l i$.

Definition 1.1.6. The map $f : U \to V$ where $U \subseteq \mathbb{C}^m, V \subseteq \mathbb{C}^n$ is *holomorphic* if it is C^1, and $j^n f_* = f_* j^m$ or equivalently the Cauchy-Riemann equations

$$\frac{\partial u^l}{\partial x^r} = \frac{\partial v^l}{\partial y^r} , \qquad \frac{\partial u^l}{\partial x^r} = -\frac{\partial v^l}{\partial y^r} ,$$

are satisfied, or equivalently f_* is of the form $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$.

Proposition 1.1.7 (Osgood's lemma). *Suppose f is holomorphic, then given $a \in U$, there is a unique power series expansion*

$$f^l(z) = \sum_{k_1, \ldots, k_m = 0}^{\infty} c_{k_1, \ldots, k_m} (z^1 - a^1)^{k_1} \cdots (z^m - a^m)^{k_m} ,$$

in some neighbourhood of a, where $a^i = z^i(a)$.

Proof. Choose $\varepsilon_1, \ldots, \varepsilon_m > 0$ such that the closure of the poly-disk Δ given by $|z^l - a^l| < \varepsilon_l$ is contained in U. For $z \in \Delta$, repeated application of the Cauchy integral formula for a single variable gives,

$$\begin{aligned} f^l(z) =& \frac{1}{(2\pi i)^m} \int_{|a^1 - \xi^1| = \varepsilon_1} \frac{\mathrm{d}\xi_1}{\xi^1 - z^1} \int_{|a^2 - \xi^2| = \varepsilon_2} \frac{\mathrm{d}\xi_2}{\xi^2 - z^2} \cdots \int_{|a^m - \xi^m| = \varepsilon_m} \frac{\mathrm{d}\xi_m}{\xi^m - z^m} f(\xi) \\ =& \frac{1}{(2\pi i)^m} \int_{|a^l - \xi^l| = \varepsilon_l} f(\xi) \frac{\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_r}{(\xi^1 - z^1) \cdots (\xi^m - z^m)} , \end{aligned}$$

where the second equality is by Fubini's theorem, since the integrand is integrable over $|a^l - \xi^l| = \varepsilon_l$. The fact the integrand is integrable over $|a^l - \xi^l| = \varepsilon_l$ follows from the fact that the integrand is bounded on $|a^l - \xi^l| = \varepsilon_l$, and $|a^l - \xi^l| = \varepsilon_l$ has a finite measure. By substituting the following series which converges absolutely uniformly on $|a^l - \xi^l| = \varepsilon_l$ into the above equality,

$$\frac{1}{(\xi^1 - z^1) \cdots (\xi^m - z^m)} = \sum_{k_1, \ldots, k_m = 0}^{\infty} \frac{(z^1 - a^1)^{k_1} \cdots (z^m - a^m)^{k_m}}{(\xi^1 - a^1)^{k_1 + 1} \cdots (\xi^m - a^m,)^{k_m + 1}} ,$$

and bringing the summation out of the integral, we get the desired expression. □

Corollary 1.1.8. *A holomorphic map $f : U \to V$ is of class C^r for every $r \in \mathbb{N}$.*

Corollary 1.1.9. *The total derivative $f_* = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ of a holomorphic map $f : U \to V$ is holomorphic as the function $A + iB : U \to M_{n \times n}(\mathbb{C}) = \mathbb{C}^{n^2}$, where $U \subseteq \mathbb{C}^m, V \subseteq \mathbb{C}^n$.*

1.2 Manifolds, vector bundles and fibre metrics

A textbook in differential geometry often treats differentiable manifolds without boundary, but waves off the treatment of manifolds with boundary and complex manifolds as being similar to manifolds without boundary. But to feel confident about the validity of these structure, one may still wants to work them out in detail. The following is an attempt to deal with these manifolds with similar structures all at once. The classical notion of a pseudo-group of transformations captures the structure of various types of manifold. We generalize it to the notation of pseudo-category of transformations, which encodes the properties of morphisms between manifolds in addition to the manifold structures.

Definition 1.2.1. A *pseudo-category of transformations* Λ on a collection $\mathcal{S}$ of topological spaces is a collection of continuous maps from an open subset of S to an open subset of S' for $S, S' \in \mathcal{S}$, such that $\mathcal{S}$ contains the singleton point space $*$ and is closed under taking product of two spaces, and

1. if $f : U \to V$ is in Λ and $W \subseteq U$ is any open subset, then $f|_W : W \to f(W)$ is in Λ;
2. (locality) given $f : U \to V$ and an open cover of U given by $U_r \subseteq U$ ranging over r, if $f|_{U_r} : U_r \to f(U_r)$ is in Λ for every r, then $f \in \Lambda$;
3. for each open subset $U \subseteq S$, $\mathrm{id}_U \in \Lambda$;
4. given maps $f : U \to V$ and $g : W \to Z$ in Λ, $f \circ g : f^{-1}(V \cap W) \to g(V \cap W)$ is in Λ (if $V \cap W = \varnothing$, we get the empty bijection, which is vacuously a homeomoprhism map between open subsets);
5. if $f \in \Lambda$ is a homeomorphism, then $f^{-1} \in \Lambda$;
6. given $f, g \in \Lambda$, $f \times g$ given by $(f \times g)(x, y) = (f(x), g(y))$ is in Λ;
7. for each open subset $U \subseteq S$, $S \in \mathcal{S}$, the unique map $*_U : U \to *$ is in Λ, and for each $a \in U$, the map $a : * \to U$ given by $a(*) = a$ is in Λ;
8. for each open subset $U \subseteq S$, $S \in \mathcal{S}$, the diagonal map $\delta_U : U \to U \times U$ given by $\delta_U(x) = (x, x)$ is in Λ;
9. given $f : U \times V \to W$, if $f(a, -), f(-, b) \in \Lambda$ for all $a \in U$, $b \in V$, then $f \in \Lambda$.

Let $\Lambda(S, S') \subseteq \Lambda$ denotes the subset of Λ consisting of maps f from an open set of S to an open set of S' for $S, S' \in \mathcal{S}$, and let $\Lambda(S) \subseteq \Lambda(S, S)$ denote the subset of $\Lambda(S, S)$ consisting of homeomorphisms for $S \in \mathcal{S}$. A *pseudo-groupoid of transformations* on a collection of topological spaces $\mathcal{S}$ (with no additional requirements) is a collection of homeomorphisms from an open subset of S to an open subset of S' for $S, S' \in \mathcal{S}$ satisfying 1 through 5. A *pseudo-group of transformations* is a pseudo-groupoid of transformations on a single topological space. In particular, each $\Lambda(S)$ above is a pseudo-group of transformations.

Remark 1.2.2. Given a pseudo-category of transformations Λ, $f \in \Lambda$, $f : U \to V$, and $W \subseteq S$, the projection $f \times *_W : U \times W \to V \times * = V$ is in Λ. Given $f \in \Lambda$ such that $f : U \times V \to W$, and $a \in U$, we have $f(a, -) = f \circ (a \times \mathrm{id}_V) : V = * \times V \to U \times V \to W$ is in Λ.

Remark 1.2.3. A pseudo-category of transformations may be viewed as a category where objects are topological spaces and morphisms are continuous maps from open subsets to open subsets, satisfying certain additional properties.

Example 1.2.4. We have the following examples:

1. The pseudo-category $\Lambda^{r,\mathbb{N}}(H,\mathbb{R})$ of C^r-transformations consisting of C^r maps between open subsets of spaces in the collection of topological spaces generated by $\mathbb{R}$ and $H = [0,\infty)$, and the associated pseudo-groups $\Lambda^r(\mathbb{R}^n)$ on $\mathbb{R}^n$, $\Lambda^r(H \times \mathbb{R}^n)$ on $H \times \mathbb{R}^n$, and $\Lambda^r(H^m \times \mathbb{R}^n)$ on $H^m \times \mathbb{R}^n$.

2. The pseudo-groupoid $\Lambda_o^{r,\mathbb{N}}(H,\mathbb{R})$ of orientation preserving transformations is the subsets of $\Lambda^{r,\mathbb{N}}(H,\mathbb{R})$ consisting of homeomorphisms f such that $\det f_*$ is positive, with associated pseudo-groups $\Lambda_o^r(\mathbb{R}^n)$, $\Lambda_o^r(H \times \mathbb{R}^n)$ and $\Lambda_o^r(H^m \times \mathbb{R}^n)$.

3. The pseudo-category $\Lambda^{\mathbb{N}}(\mathbb{C})$ of holomorphic transformations consisting of holomorphic maps from an open subset of $\mathbb{C}^n$ to an open subset of $\mathbb{C}^m$, and the associated pseudo-groups $\Lambda(\mathbb{C}^n)$ on $\mathbb{C}^n$.

Remark 1.2.5. By Corollary 1.1.8 and Definition 1.1.6, we have $\Lambda(\mathbb{C}^n) \subseteq \Lambda_o^{r,\mathbb{N}}(H,\mathbb{R}) \subseteq \Lambda^{r,\mathbb{N}}(H,\mathbb{R})$.

Definition 1.2.6. Given topological spaces S, M, and a pseudo-group of transformations Λ on S, a Λ-*atlas of* M is a family of pairs (U_i, φ_i), called Λ-*charts*, indexed over a set I, such that U_i indexed over I is an open cover of M, and

1. for every $i \in I$, $\varphi_i : U_i \to V_i$ is a homeomprhism, where $V_i \subseteq S$ is an open subset;

2. for every $i, j \in I$, $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is an element of Λ.

A Λ-*structure of* M is a Λ-atlas of M that is not a proper subfamily of any Λ-atlases of M. A Λ-*space* is a topological space equipped with a Λ-structure. A Λ-*manifold* is a second countable Hausdorff Λ-space.

Remark 1.2.7. Given an open subset $U \subseteq M$, a Λ-atlas of M gives an unique *induced* Λ-*atlas* on U.

Proposition 1.2.8. *Given a Λ-atlas A of M, there exists a unique Λ-structure of M containing A as a subfamily.*

Proof. Let $\tilde{A}$ be the family of pairs (U, φ) such that $\varphi : U \to V$ is a homeomoprhism, $V \subseteq S$ is an open subset, and $\varphi_i \circ \varphi^{-1} : \varphi(U_i \cap U) \to \varphi_i(U_i \cap U)$ is an element of Λ for every (U_i, φ_i) in A. Thus $A \subseteq \tilde{A}$. For any chart (U, φ) in some Λ-atlas A' containing A, we get $(U, \varphi) \in \tilde{A}$, so $\tilde{A}$ is not properly contained in any Λ-atlas. We check that $\tilde{A}$ is an Λ-atlas. Since $A \subseteq \tilde{A}$, $\tilde{A}$ covers M by open sets. Given (U, φ) and (W, ψ) in $\tilde{A}$, and (U_i, φ_i) in A, we have $\psi \circ \varphi_i, \varphi \circ \varphi_i \in \Lambda$, and the map $\psi \circ \varphi^{-1} : \varphi(U \cap W \cap U_i) \to \psi(U \cap W \cap U_i)$ is equal to $\psi \circ \varphi_i \circ (\varphi \circ \varphi_i)^{-1} \in \Lambda$. Since $\varphi(U \cap W) = \bigcup_i \varphi(U \cap W \cap U_i)$, we get $\psi \circ \varphi^{-1} : \varphi(U \cap W) \to \psi(U \cap W)$ is in Λ by locality of Λ. Hence $\tilde{A}$ is a Λ-structure. $\square$

Corollary 1.2.9 (Manifold chart lemma)**.** *Let S be a topological space, and Λ a pseudo-group of transformations on S. Given a set M, a family (U_i, φ) indexed over some I, where $U_i \subseteq M$ ranging over a countable subset of I covers M, $\varphi_i : U_i \to V_i$ are bijections such that V_i and $\varphi_i(U_i \cap U_j)$ are open subsets of S, and $\varphi_j \circ \varphi_i^{-1} \in \Lambda$ for all $i, j \in I$.*

If M is given the minimal topology making φ_i continuous for all $i \in I$, then there is an unique Λ-structure of M containing the pairs (U_i, φ_i) as charts. If additionally S is second-countable and any $p, q \in M$ has $i, j \in I$ such that $p \in U_i$, $q \in U_j$ and $U_i \cap U_j = \varnothing$, then M is second countable Hausdorff.

Proof. By minimality, each φ_i is a homeomorphism, so the conclusion follows from Proposition 1.2.8. Suppose the additional assumptions hold. Since each φ_i is a homeomorphism, U_i is open, thus M is Hausdorff. Also, U_i is second countable, where M is covered by countably many U_i, hence M is second countable. $\square$

Definition 1.2.10. Let Λ be a pseudo-category of transformations, and A and B be a $\Lambda(S)$-structure and a $\Lambda(S')$-structure on topological spaces M and N respectively. A continuous map $f : M \to N$ is a Λ-*map* if for any charts $(U, \varphi) \in A$ and $(V, \psi) \in B$, we have $\psi \circ f \circ \varphi^{-1} \in \Lambda(S, S')$. If f is a Λ-homeomorphism, and f^{-1} is a Λ-map, then f is a Λ-*isomorphism.*

Remark 1.2.11. The product $M \times N$ can be given the $\Lambda(S \times S')$-atlas $A \times B = \{(U \times V, \varphi \times \psi) : (U, \varphi) \in A, (V, \psi) \in B\}$. Then for two Λ-maps (Λ-isomorphisms) $f : M \to N$, $g : M' \to N'$, the product map $f \times g : M \times M' \to N \times N'$ is also a Λ-map (resp. Λ-isomorphism).

Definition 1.2.12. A *differentiable manifold of class C^r (resp. differentiable manifold of class C^r with boundaries, differentiable manifold of class C^r with corners, oriented differentiable manifold of class C^r, oriented differentiable manifold of class C^r with boundaries, oriented differentiable manifold of class C^r with corners, complex manifold) of dimension n* is a $\Lambda^r(\mathbb{R}^n)$-manifold (resp. $\Lambda^r(H \times \mathbb{R}^n)$-manifold, $\Lambda^r(H^m \times \mathbb{R}^n)$-manifold, $\Lambda^r_o(\mathbb{R}^n)$-manifold, $\Lambda^r_o(H \times \mathbb{R}^n)$-manifold, $\Lambda^r_o(H^m \times \mathbb{R}^n)$-manifold, $\Lambda(\mathbb{C}^n)$-manifold). A *C^r-differentiable map (resp. holomorphic map, C^r-diffeomorphism, orientation preserving C^r-diffeomorphism, biholomorphic map)* is a $\Lambda^{r,\mathbb{N}}(H, \mathbb{R})$-map (resp. $\Lambda^{\mathbb{N}}(\mathbb{C})$-map, $\Lambda^{r,\mathbb{N}}(H, \mathbb{R})$-isomorphism, $\Lambda^{r,\mathbb{N}}_o(H, \mathbb{R})$-isomorphism, $\Lambda^{\mathbb{N}}(\mathbb{C})$-isomorphism).

Remark 1.2.13. By Remark 1.2.5, a complex manifold is an oriented differeiantable manifold of class C^∞.

Remark 1.2.14. Let $x^1, \ldots, x^n$ (resp. $z^1, \ldots, z^n$) be the standard coordinates for $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), and let (U, φ) be a C^r-differentiable (resp. holomorphic) chart of real dimension n (resp. $2n$). We abuse notation and write $x^i = x^i \circ \varphi$ (resp. $z^i = z^i \circ \varphi$), and call these *local C^r-coordinates* (resp. *local holomorphic coordinates*) associated to (U, φ).

Definition 1.2.15. A *C^k function* (resp. *homomorphic function*) on an open subset U of a differentiable manifold of class C^r (resp. complex manifold) M is a C^r map (resp. holomorphic map) $f : U \to \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$). A *C^k curve* passing $x \in M$ at $t \in I$ is a C^k-map $\gamma : I \to M$ with $\gamma(t) = x$, where $I \subseteq \mathbb{R}$ is an open interval. A *holomorphic curve* passing $x \in M$ at $z \in U$ is a holomorphic map $\gamma : U \to M$ with $\gamma(z) = x$, where $U \subseteq \mathbb{C}$ is an open connected set.

Definition 1.2.16. Let Λ be a pseudo-category of transformations on a collection of topological spaces containing a topological field $\mathbb{K}$ such that addition and scalar multiplication of $\mathbb{K}$ are in Λ, and M be a $\Lambda(S)$-space. A *$\mathbb{K}$-vector bundle of rank r* over M is a topological space E equipped with an $\Lambda(S \times \mathbb{K}^r)$-atlas, a Λ-surjection $\pi : E \to M$ called the *projection map*, and a r-dimensional $\mathbb{K}$-vector space structure on $E_x = \pi^{-1}(x)$, called the *fibre at* x, for each $x \in M$, such that for every $p \in M$ there exists an open subset $U \ni p$ and a Λ-isomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{K}^n$, called a *local trivialization*, such that $\Phi|_{E_x}$ is a linear isomorphism to $\{x\} \times \mathbb{K}^n$ for each $x \in U$.

Remark 1.2.17. Since $\mathbb{K} \in \mathcal{S}$, and addition and scalar multiplication on $\mathbb{K}^n$ are in Λ, the map $M_{m\times n}(\mathbb{K}) \times \mathbb{K}^n \to \mathbb{K}^m$ given by $(A, v) \mapsto Av$ is in Λ, where $M_{m\times n}(\mathbb{K}) = \mathbb{K}^{mn}$.

Remark 1.2.18. For local trivializations (U_i, Φ_i) and (U_j, Φ_j), the Λ-isomorphism $\Phi_j \circ \Phi_i^{-1}$ has the form $(\Phi_j \circ \Phi_i^{-1})(p, v) = (p, g_{ij}(p)v)$, where $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{K}) \subseteq \mathbb{K}^{n^2}$ are Λ-maps called *transition maps*.

Definition 1.2.19. Given a $\mathbb{K}$-vector bundle E of rank r over a $\Lambda(S)$-space M, and an open subset $U \subseteq E$, a Λ-*section* of E over U is a Λ-map $s : U \to M$ such that $\pi \circ s = \mathrm{id}_U$. The set of Λ-sections of E over U is denoted $\Gamma(E, U)$, and $\Gamma(E) = \Gamma(E, M)$.

Definition 1.2.20. Let E, F be $\mathbb{K}$-vector bundles over a $\Gamma(S)$-space M and $\Gamma(S')$-space N with projection maps π_1, π_2 respectively. A *vector bundle morphism* $f : E \to F$ is a Γ-map such that there exists a Γ-map $g : M \to N$ *covered by* f, meaning $\pi_2 \circ f = g \circ \pi_1$, and $f|_{E_x}$ is a $\mathbb{K}$-linear map for each $x \in M$. The *rank (resp. nullity) of* f *at* $x \in M$ is the rank (resp. nullity) of $f|_{E_x}$.

Definition 1.2.21. A rank r subbundle of E is a $\mathbb{K}$-vector bundle F of rank r with a vector bundle morphism $\iota : F \to E$ covering id_M with constant rank r and nullity 0.

Proposition 1.2.22 (Vector bundle chart lemma)**.** *Let* Λ, $\mathbb{K}$, *and* M *be as above, and let* E_x *be a* r*-dimensional* $\mathbb{K}$*-vector space for each* $x \in M$*. Given a family* (U_i, Φ_i) *indexed over some* I*, such that* U_i *ranging over* I *covers* M *with open sets, and* $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^r$ *are bijections,* $\Phi_i|_{E_x}$ *is linear,* $\pi : E \to M$ *is defined by sending* E_x *to* x*, with* $E = \bigsqcup_{x\in M} E_x$*, where* $\Phi_j \circ \Phi_i^{-1}$ *is a* Λ *map. If* E *is given the minimal topology making* Φ_i *continuous for each* $i \in I$*, there is a unique* $\Lambda(S \times \mathbb{K}^r)$*-structure making* E *a* $\mathbb{K}$*-vector bundle of rank* r *over* M*. If in addition* M *is second countable Hausdorff, then so is* E*.*

Proof. For each $x \in M$, there is a (U_i, Φ_i) and a chart (V_x, φ_x) in the $\Lambda(S)$-structure on M such that $U_i \supseteq V_x \ni x$. Then the composition $\psi_x = (\varphi_x \times \mathrm{id}_{\mathbb{K}^r}) \circ \Phi_i : \pi^{-1}(V_x) \to \varphi_x(V_x) \times \mathbb{K}^r \subseteq S \times \mathbb{K}^r$ is a bijection onto an open subset. For $y \in M$, let $\psi_y = (\varphi_y \times \mathrm{id}_{\mathbb{K}^r}) \circ \Phi_j$, then $\psi_x(\pi^{-1}(V_x) \cap \pi^{-1}(V_y)) = \psi_x(\pi^{-1}(V_x)) \cap (\varphi_x(V_y) \times \mathbb{K}^r) = \varphi_x(V_x \cap V_y) \times \mathbb{K}^r$, which is open, and $\psi_y \circ \psi_x^{-1} = (\varphi_y \times \mathrm{id}_{\mathbb{K}^r}) \circ \Phi_j \circ \Phi_i^{-1} \circ (\varphi_x \times \mathrm{id}_{\mathbb{K}^r})^{-1} \in \Lambda$. By Corollary 1.2.9, E with the minimal topology making the Φ_i continuous has a unique $\Lambda(S \times \mathbb{K}^r)$-structure containing each (V_x, ψ_x), and if M is second countable Hausdorff, so is E. Since $\varphi_x \circ \pi \circ \psi_x^{-1} : \varphi_x(V_x) \times \mathbb{K}^r \to \varphi_x(V_x)$ is equal to $\mathrm{id}_{\varphi_x(V_x)} \times *_{\mathbb{K}^r}$, π is a Λ-surjection by Remark 1.2.2, and $\varphi_x \circ \Phi_i \circ \psi_x^{-1} = \mathrm{id}_{\varphi_x(V_x)\times\mathbb{K}^r}$, so Φ_i are Λ-isomorphisms. □

Remark 1.2.23. The condition $\Phi_j \circ \Phi_i^{-1}$ is a Λ-map is met if there is a Λ-map $g_{ij} : U_i \cap U_j \to GL(n, \mathbb{K})$ such that $\Phi_j \circ \Phi_i^{-1}(p, v) = (p, g_{ij}(p)v)$.

Corollaries 1.2.24 and 1.2.28 below follow from Proposition 1.2.22.

Corollary 1.2.24. *Let* E *and* F *be* $\mathbb{K}$*-vector bundles over* M*. We have* $\mathbb{K}$*-vector bundles* E^*, $E \oplus F$, $E \otimes F$, $\bigwedge^n E$, *and* $\mathrm{Sym}^n E$ *over* M*, with fibres* E_x^*, $E_x \oplus F_x$, $E_x \otimes F_x$, $\bigwedge^n E_x$, *and* $\mathrm{Sym}^n E_x$*, and given sections* $s, s_1, \ldots, s_n \in \Gamma(E, U)$ *and* $t \in \Gamma(F, U)$, $s^* \in \Gamma(E^*, U)$, $s \oplus t \in \Gamma(E \oplus F, U)$, $s \otimes t \in \Gamma(E \otimes F, U)$, $s_1 \wedge \cdots \wedge s_n \in \Gamma(\bigwedge^n E, U)$, *and* $s_1 \odot \cdots \odot s_n \in \Gamma(\mathrm{Sym}^n E, U)$ *respectively. If* $\mathbb{K} = \mathbb{C}$ *has a conjugation* $a \mapsto \bar{a}$ *which is a field automorphism, we also have* $\bar{E}$ *with fibres* $\bar{E}_x$*, where* $\bar{E}_x$ *is a copy of* E_x *with elements denoted* $\bar{v}$ *for* $v \in E_x$ *such that* $z \cdot \bar{v} = \overline{\bar{z} \cdot v}$ *for* $z \in \mathbb{C}$*.*

Remark 1.2.25. If $\mathbb{K}$ is a finite field extension of $\mathbb{L}$, E is a $\mathbb{K}$-vector bundle, and F is a $\mathbb{L}$-vector bundle, then $E \otimes F$ is naturally a $\mathbb{K}$-vector bundle.

Remark 1.2.26. A set of sections $s_1, \dots, s_r \in \Gamma(E, U)$ is a *local frame* of E over U if it is a basis at each $x \in U$, and the set of dual sections $s^1, \dots, s^1 \in \Gamma(E^*, U)$ are the associated *local coordinates*.

Remark 1.2.27. A section $s \in \Gamma(E^* \otimes E)$ is a vector bundle morphism $s : E \to E$ covering id_M, for a $\mathbb{K}$-vector bundle E over M. In particular, the λ-eigenspaces of s for $\lambda \in \mathbb{K}$ form a subbundle of E if s has constant λ-geometric multiplicity.

Corollary 1.2.28. *Given a Λ-map $f : M \to N$, and a $\mathbb{K}$-vector bundle E over N, we have the pullback bundle $f^*E = \bigsqcup_{x \in M} E_{f(x)}$ over M, such that $g : f^*E \to E$ which is the identity on each fibre is a $\mathbb{K}$-vector bundle morphism covering f.*

Definition 1.2.29. A *Riemannian metric* (resp. *Hermitian metric*) on a $\mathbb{R}$-vector bundle (resp. $\mathbb{C}$-vector bundle) E is a section $h \in \Gamma(E^* \otimes E^*)$ (resp. $\Gamma(E^* \otimes \bar{E}^*)$) such that $h(v, v) > 0$ for all $v \neq 0$ and $h(v, w) = h(w, v)$ (resp. $h(v, w) = \overline{h(w, v)}$).

1.3 Cotangent (tangent) bundle and connections

Tangent vectors are often defined as operators on the space of smooth functions satisfying the product rule, notably in Lee [1], and cotangent vectors are defined as the dual. This definition is simple, but somewhat abstract is the sense that it is harder to visualize cotangent vectors geometrically. This section uses a direct construction of the cotangent bundle using sheaf theory, defining cotangent spaces as the quotient of the space of germs of differentiable functions by germs of functions constant to first order at the point. The approach adds complications but has the advantage of being more intuitive.

Sheaf theory

Only the theory essential for the construction of the cotangent bundle is presented. The interested reader may consult [8] for a more elaborate treatment.

For our purposes, a ring will be a commutative ring.

Definition 1.3.1. A *presheaf of rings (resp. abelian groups)* $\mathcal{F}$ on a topological space M consists of:

1. for each open subset $U \subseteq M$, a ring (resp. abelian group) $\mathcal{F}(U)$;
2. for each pair $V \subseteq U \subseteq M$ of open subsets, a ring (resp. group) homomorphism $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$, called the *restriction map*, such that $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$, and $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ for open subsets $W \subseteq V \subseteq U \subseteq M$.

Let $\mathcal{A}$ be a presheaf of rings on a topological space M, with restriction maps $\mathrm{res}_{U,V}$. A *presheaf of $\mathcal{A}$-modules* on M is a presheaf of abelian groups $\mathcal{F}$ on M such that $\mathcal{F}(U)$ is a $\mathcal{A}(U)$-module for each open subset $U \subseteq S$, and the restriction maps $\rho_{U,V}$ satisfy $\rho_{U,V}(a \cdot s) = \mathrm{res}_{U,V}(a) \cdot \rho_{U,V}(s)$ for all $a \in \mathcal{A}(U)$ and $s \in \mathcal{F}(U)$ for open subsets $V \subseteq U \subseteq M$.

Definition 1.3.2. Let $\mathcal{A}$ be a presheaf of rings on a topological space M and $\mathcal{F}$ a presheaf of $\mathcal{A}$-modules with restriction maps $\rho_{U,V}$. A presheaf of $\mathcal{A}$-modules $\mathcal{G}$ is a *sub-presheaf* of $\mathcal{A}$-modules of $\mathcal{F}$ if $\mathcal{G}(U)$ is a $\mathcal{A}(U)$-submodule of $\mathcal{F}(U)$ and $\rho_{U,V}(\mathcal{G}(U)) \subseteq \mathcal{G}(V)$ for every open subsets $V \subseteq U \subseteq M$ where the restriction maps of $\mathcal{G}$ are $\rho_{U,V}|_{\mathcal{G}(U)} : \mathcal{G}(U) \to \mathcal{G}(V)$.

Definition 1.3.3. Let $\mathcal{A}$ be a presheaf of rings on a topological space M and $\mathcal{F}, \mathcal{G}$ be presheaves of $\mathcal{A}$-modules with restriction maps $\rho_{U,V}, \rho'_{U,V}$ respectively. The *direct sum* $\mathcal{F} \oplus \mathcal{G}$ is the presheaf of $\mathcal{A}$-modules such that $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ with the restriction maps being $\rho_{U,V} \oplus \rho'_{U,V}$ given by $s \oplus t \mapsto \rho_{U,V}(s) \oplus \rho'_{U,V}(t)$. The *tensor product* $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is the presheaf of $\mathcal{A}$-modules such that $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{A}(U)} \mathcal{G}(U)$ with the restriction maps being $\rho_{U,V} \otimes \rho'_{U,V}$ given by $s \otimes t \mapsto \rho_{U,V}(s) \otimes \rho'_{U,V}(t)$. If $\mathcal{G}$ is a sub-presheaf of $\mathcal{A}$-modules of $\mathcal{F}$, the *quotient sheaf* $\mathcal{F}/\mathcal{G}$ is the presheaf of $\mathcal{A}$-modules such that $(\mathcal{F}/\mathcal{G})(U) = \mathcal{F}(U)/\mathcal{G}(U)$ and the restriction maps being $\tilde{\rho}_{U,V}$ given by $s + \mathcal{G}(U) \mapsto \rho_{U,V}(s) + \mathcal{G}(V)$.

Definition 1.3.4. A *directed system of rings (resp. abelian groups)* is a index family $\{X_i\}_{i \in I}$ of rings (resp. abelian groups) with a partial-order $\leq$ on I such that for each $i, j \in I$ there is $k \in I$ such that $i \leq k$ and $j \leq k$, and a ring (resp. group) homomorphism $f_{ij} : X_i \to X_j$ for each $i, j \in I$ with $i \leq j$, such that $f_{ii} = \mathrm{id}_{X_i}$, and $f_{ik} = f_{jk} \circ f_{ij}$ for $i, j, k \in I$ with $i \leq j \leq k$. The *direct limit* of the directed system of rings (resp. abelian groups), denoted $\varinjlim_{i \in I} X_i$, is the ring (resp, abelian group) with underlying set $\bigsqcup_{i \in I} X_i / \sim$ where $\sim$ the the equivalence relation given by $x_i \sim x_j$, where $x \in X_i$ and $x_j \in X_j$ for some $i, j \in I$, if and only if there is $k \in I$ such that $i \leq k$ and $j \leq k$, and $f_{ik}(x_i) = f_{jk}(x_j)$, with addition given by $[x_i]_\sim + [x_j]_\sim = [f_{ik}(x_i) + f_{jk}(x_j)]_\sim$, and multiplication given by $[x_i]_\sim [x_j]_\sim = [f_{ik}(x_i) f_{jk}(x_j)]_\sim$ for $x_i \in X_i$, $x_j \in X_j$, and $i \leq k$, $j \leq k$, $i, j, k \in I$. For $i \in I$, the *canonical map* $f_i : X_i \to \varinjlim_{i \in I} X_i$ is $f_i(x_i) = [x_i]_\sim$.

Proposition 1.3.5. *The ring (resp. group) structure on $\varinjlim_{i \in I} X_i$ is well-defined and gives a ring (resp. abelian group), and the canonical maps f_i are ring (resp. group) homomorphisms.*

Proof. Let $x_i \in X_i$, $x_j \in X_j$, $x_k \in X_j$, $x_l \in X_l$ such that there is $r, s \in I$, $i, j \leq r$, $k, l \leq s$, such that $f_{ir}(x_i) = f_{jr}(x_j)$ and $f_{ks}(x_k) = f_{ls}(x_l)$, then there is $t \in I$, $r, s \leq t$, so $f_{it}(x_i) f_{kt}(x_k) = f_{rt}(f_{ir}(x_i)) f_{st}(f_{ks}(x_k)) = f_{rt}(f_{jr}(x_j)) f_{st}(f_{ls}(x_l)) = f_{jt}(x_j) f_{lt}(x_l)$, so the multiplication is well-defined. Addition is checked similarly. The other ring (resp. abelian group) axioms follow from X_i being rings (resp. abelian groups) and the f_{ij} being ring (resp. group) homorphisms. Given $x_i, y_i \in X_i$, we have $[x_i y_i]_\sim = [f_{ii}(x_i) f_{ii}(y_i)]_\sim = [x_i]_\sim [y_i]_\sim$, and similarly for addition, so f_i is a ring (resp. group) homomorphism. $\square$

Definition 1.3.6. Given a presheaf $\mathcal{F}$ of rings (resp. abelian groups) on a topological space M, the *stalk* of $\mathcal{F}$ at $x \in M$ is the direct limit $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$ of the directed system consisting of $\mathcal{F}(U)$ over open subsets $U \ni x$ of M, with the partial-order $\supseteq$ and the restriction maps. Denote $s_x = [s]_\sim$ for $s \in \mathcal{F}(U)$. Elements of the form s_x are sometimes called *germs* at x.

Proposition 1.3.7. *For a presheaf of rings $\mathcal{A}$ on a topological space M and a presheaf of $\mathcal{A}$-modules $\mathcal{F}$ on M, $\mathcal{F}_x$ is an $\mathcal{A}_x$-module by the action $a_x \cdot m_x = \rho_W(res_{U,W}(a) \cdot \rho_{V,W}(m))$ for $a \in \mathcal{A}(U)$, $m \in \mathcal{F}(V)$, where $U, V \supseteq W \ni x$ are open subsets of M, and ρ_W satisfies $\rho_W(a \cdot s) = res_W(a) \cdot \rho_W(s)$ for all $a \in \mathcal{A}(W)$ and $s \in \mathcal{F}(W)$, where res_W, ρ_W are the canonical maps.*

Proof. Similar to Proposition 1.3.5. □

Proposition 1.3.8. *Given a presheaf of rings $\mathcal{A}$ on a topological space M, a presheaf of $\mathcal{A}$-modules $\mathcal{F}$, and a sub-presheaf of $\mathcal{A}$-modules $\mathcal{G}$ of $\mathcal{F}$, $\mathcal{G}_x$ is a $\mathcal{A}_x$-submodule of $\mathcal{F}_x$ for each $x \in M$.*

Proof. Let ρ_W be the canonical maps to $\mathcal{F}_x$, then $\rho_W(\mathcal{G}(W)) = \mathcal{G}_x$ for each $W \ni x$, so $\mathcal{G}_x$ is a submodule of $\mathcal{F}_x$ by Proposition 1.3.5 and Proposition 1.3.7. □

Proposition 1.3.9. *Given a presheaf of rings $\mathcal{A}$ on a topological space M, and presheaves of $\mathcal{A}$-modules $\mathcal{F}, \mathcal{G}$, $(\mathcal{F} \oplus \mathcal{G})_x = \mathcal{F}_x \oplus \mathcal{G}_x$ and $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$, and if $\mathcal{G}$ is a sub-presheaf of $\mathcal{A}$-modules of $\mathcal{F}$, then $(\mathcal{F}/\mathcal{G})_x = \mathcal{F}_x/\mathcal{G}_x$.*

Proof. We write $s \sim t$ for the relation that there is some restriction map which maps s, t to the same element. By the definition, we have $s \oplus s' \sim t \oplus t'$ if and only if $s \sim t$ and $s' \sim t'$, so we may identify $(\mathcal{F} \oplus \mathcal{G})_x = \mathcal{F}_x \oplus \mathcal{G}_x$, with the canonical maps $\rho_W \oplus \rho'_W$, which gives the $\mathcal{A}_x$-module structure. Similarly, $s \otimes_{\mathcal{A}(U)} s' \sim t \otimes_{\mathcal{A}(V)} t'$ if and only if $s \sim t$ and $s' \sim t'$, so we may identify $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$, with canonical maps $\rho_W \otimes \rho'_W$. We have $s + \mathcal{G}(U) \sim t + \mathcal{G}(V)$ if and only if $\rho_{U,W}(s) - \rho_{V,W}(t) \in \mathcal{G}(W)$ for some $W \subseteq U, V$, so we may identify $(\mathcal{F}/\mathcal{G})_x = \mathcal{F}_x/\mathcal{G}_x$, with canonical map $\mathcal{F}(W)/\mathcal{G}(W) \to \mathcal{F}_x/\mathcal{G}_x$ induced from $\rho_W : \mathcal{F}(W) \to \mathcal{F}_x$. □

Example 1.3.10. A presheaf $\mathcal{F}$ may be defined with $\mathcal{F}(U)$ consisting of functions on U, with the restriction maps $\rho_{U,V}$ given by sending f to $f|_V$. Some examples are:

1. The *presheaf of C^r-functions* $C^r(-)$ on a differentiable manifold of class C^k, where $r \leq k$, $r, k \in \mathbb{N}^+ \cup \{\infty\}$.
2. The *presheaf of holomorphic functions* $\mathcal{H}(-)$ on a complex manifold.
3. The *presheaf of constant functions* $\mathbb{R}(-)$ on a differentiable manifold of class C^r, or $\mathbb{C}(-)$ on a complex manifold.

Cotangent bundle

When constructing structures on differentiable manifolds, the property of being r-times differentiable is often unstable. For example, the tangent bundle of a C^r manifold is C^{r-1}, and a differentiable vector field sends C^r functions to C^{r-1} functions. Therefore we will only work with the C^∞ case eventually, although some results will still be stated in general when permitted.

Proposition 1.3.11. *Given a differentiable manifold of class C^r (resp. complex manifold) M where $r \geq 0$, the stalk C^r_x (resp. $\mathcal{H}_x$) is a $\mathbb{R}$-vector space (resp. $\mathbb{C}$-vector space) for each $x \in M$.*

Proof. Given $x \in M$, the sheaf of rings $\mathbb{R}(-)$ with restriction maps $\mathrm{res}_{U,V}$ has $\mathrm{res}_{U,W}(f) = \mathrm{res}_{V,W}(g)$ for $f \in \mathbb{R}(U)$, $g \in \mathbb{R}(V)$, and $U, V \supseteq W \ni x$, if and only if $f(x) = g(x)$, so the stalk of $\mathbb{R}(-)$ at x is $\mathbb{R}$. The presheaf $C^r(-)$ is a presheaf of $\mathbb{R}(-)$-modules, so C^r_x is a $\mathbb{R}$-vector space by Proposition 1.3.7. Similarly for $\mathcal{H}_x$. □

Proposition 1.3.12. *The subset $\mathcal{K}$ of C^r_x consisting of those $f_x \in C^\infty_x$, where $f \in C^\infty(U)$ with $x \in U$, such that $\frac{\mathrm{d} f \circ \gamma}{\mathrm{d}t}|_{t=0} = 0$, for all C^1 curves γ passing x at 0, where $r \geq 1$, is a $\mathbb{R}$-subspace. The subset $\mathcal{L}$ of $\mathcal{H}_x$ consisting of those $f_x \in \mathcal{H}_x$, where $f \in \mathcal{H}(U)$ with $x \in U$, such that $\frac{\mathrm{d} f \circ \gamma}{\mathrm{d}z}|_{z=0} = 0$, for all holomorphic curves γ passing x at 0, is a $\mathbb{C}$-subspace.*

Proof. The functional $\frac{\mathrm{d}(-)\circ\gamma}{\mathrm{d}t}|_{t=0}$ on C_x^r is well-defined and $\mathbb{R}$-linear, for a fixed C^1 curve γ passing x at 0, so its kernel $\mathcal{K}_\gamma$ is a $\mathbb{R}$-subspace. As the intersection of $\mathcal{K}_\gamma$ over all γ, $\mathcal{K}$ is a $\mathbb{R}$-subspace. The case for $\mathcal{L}$ is similar. $\square$

Definition 1.3.13. The *cotangent space* at $x \in M$ of a differentiable manifold M of class C^r is the quotient $\mathbb{R}$-vector space $T_x^*M = C_x^r/\mathcal{K}$. The *holomorphic cotangent space* at $x \in M$ of a complex manifold M is the quotient $\mathbb{C}$-vector space $H_x^*M = \mathcal{H}_x/\mathcal{L}$. Denote $\mathrm{d}f_x = f_x + \mathcal{K}$ (resp. $\mathrm{d}f_x = f_x + \mathcal{L}$) for $f \in C^r(U)$ (resp. $f \in \mathcal{H}(U)$) where $U \ni x$ is an open subset of M.

Proposition 1.3.14. *There is a natural $\mathbb{C}$-linear inclusion $H_x^*M \hookrightarrow T_x^*M \otimes_{\mathbb{R}} \mathbb{C}$ for a complex manifold M given by $\mathrm{d}f_x \mapsto \mathrm{d}u_x + i\mathrm{d}v_x$ where $f = u + iv$.*

Proof. Since $H_x^*M = \mathcal{H}_x/\mathcal{L}$, and $C^\infty(U) \otimes \mathbb{C}$ quotients to $T_x^*M \otimes \mathbb{C}$ by taking the stalk at x then modding out $\mathcal{K} \otimes \mathbb{C}$, the inclusion $\mathcal{H}(U) \subseteq C^\infty(U) \otimes \mathbb{C}$ induces a linear injection $H_x^*M \hookrightarrow T_x^*M \otimes_{\mathbb{R}} \mathbb{C}$. Specifically, the identification $f \mapsto u + iv$ gives a natural inclusion $\mathcal{H}(U) \subseteq C^\infty(U) \otimes \mathbb{C}$ for each $U \ni x$, realizing $\mathcal{H}(-)$ as a sub-presheaf of the presheaf of $\mathbb{C}(-)$-modules $C^\infty(-) \otimes_{\mathbb{R}(-)} \mathbb{C}(-)$, which passes to a $\mathbb{C}$-linear inclusion $\mathcal{H}_x \hookrightarrow C_x^\infty \otimes \mathbb{C}$ given by $f_x \mapsto u_x + iv_x$. Since $\mathcal{L} \subseteq \mathcal{K} \otimes_{\mathbb{R}} \mathbb{C}$, we are able to get a $\mathbb{C}$-linear map $H_x^*M \to T_x^*M \otimes_{\mathbb{R}} \mathbb{C}$ between the quotients, given by $\mathrm{d}f_x \mapsto \mathrm{d}u_x + i\mathrm{d}v_x$, where $\mathrm{d}u_x = u_x + \mathcal{K} \otimes_{\mathbb{R}} \mathbb{C}$ and similarly for $\mathrm{d}v_x$. Suppose $f = u + iv \in \mathcal{H}(U)$ and $u_x, v_x \in \mathcal{K}$, then $f \in \mathcal{L}$ necessarily, so the above map is injective. Another way to say this is $\mathcal{L} = \mathcal{H}_x \cap (\mathcal{K} \otimes_{\mathbb{R}} \mathbb{C})$. $\square$

Proposition 1.3.15. *Given $f_x^1, \dots, f_x^k \in C_x^r$ (resp. $\mathcal{H}_x$), and a C^r (resp. holomorphic) function g on U, where $r \geq 1$, $U \ni y$ is an open subset of $\mathbb{R}^k$ (resp. $\mathbb{C}^k$) and $y = (f^1(x), \dots, f^k(x))$, there is a well-defined $h_x = g(f^1, \dots, f^k)_x \in C_x^r$ (resp. $\mathcal{H}_x$) such that $\mathrm{d}h_x = \frac{\partial g}{\partial f^l}(y)\mathrm{d}f_x^l$, with $\frac{\partial g}{\partial f^l}$ denoting the partial derivative of g in the l-th variable.*

Proof. Let $f^1, \dots, f^k : W \to \mathbb{R}$ such that f_x^l are the germs we are given. Define $h(p) = g(f^1(p), \dots, f^k(p))$ on some $V \ni x$, then $h_x \in C_x^r$. For every C^1 curve γ passing x at 0, the chain rule states $\frac{\mathrm{d}h\circ\gamma}{\mathrm{d}t}|_{t=0} = \frac{\partial g}{\partial f^l}(y)\frac{\mathrm{d}f^l\circ\gamma}{\mathrm{d}t}|_{t=0}$, so by linearity we have $h_x - \frac{\partial g}{\partial f^l}(y)f_x^l \in \mathcal{K}_\gamma$. Thus $h_x - \frac{\partial g}{\partial f^l}(y)f_x^l \in \mathcal{K}$. The holomorphic case is verbatim using the chain rule for holomorphic maps. $\square$

Corollary 1.3.16 (Product rule)**.** *Given $f_x, g_x \in T_x^*M$ (resp. H_x^*M), we have $\mathrm{d}(fg)_x = g(x)\mathrm{d}f_x + f(x)\mathrm{d}g_x$.*

Corollary 1.3.17. *If M is a n-dimensional differentiable manifold of class C^r (resp. $2n$-dimensional complex manifold), where $r \geq 1$, given a C^r-chart (resp. holomorphic chart) (U, φ) with $U \ni p$, let $x^1, \dots, x^n$ (resp. $z^1, \dots, z^n$) be the associated local coordinates (resp. local holomorphic coordinates), then $\mathrm{d}x_p^1, \dots, \mathrm{d}x_p^n$ (resp. $\mathrm{d}z_p^1, \dots, \mathrm{d}z_p^n$) form a basis for T_p^*M (resp. H_p^*M), so we get bijections $\Phi_U : E_U \to U \times \mathbb{K}^r$ for $E_U = \bigsqcup_{x\in U} T^*M$ and $\mathbb{K} = \mathbb{R}$ (resp. $E_U = \bigsqcup_{x\in U} H^*M$ and $\mathbb{K} = \mathbb{C}$) by sending $(p, \mathrm{d}f)$ to $(p, \frac{\partial f}{\partial x^i}e_i)$ (resp. $(p, \frac{\partial f}{\partial z^i}e_i)$), such that $\Phi_V \circ \Phi_U^{-1}$ are C^{r-1}-maps (resp. holomorphic maps).*

Proof. As a direct consequence of Proposition 1.3.15, this is a spanning set. Given c_i such that $c_i x_p^i \in \mathcal{K}$, where x_p^i are germs, we have $\frac{\mathrm{d}(c_i x^i\circ\gamma)}{\mathrm{d}t}|_{t=0} = 0$ for any C^1 curve passing 0 at p. Let e_i be the standard basis on $\mathbb{R}^n$ and let $\gamma_k(t) = \varphi^{-1}(x^i(p)e_i + te_k)$, where (U, φ) is the chart giving the local coordinates. Then $0 = \frac{\mathrm{d}(c_i x_p^i\circ\gamma_k)}{\mathrm{d}t}|_{t=0} = c_k$. So it is linearly independent. The same goes for the holomorphic case. $\square$

Remark 1.3.18. From Corollary 1.3.17, $\dim T_x^*M = \dim M$ and $2\dim_{\mathbb{C}} H_x^*M = \dim M$. So given a complex manifold M, the vector space H_x^*M has half the $\mathbb{C}$-dimension of $T_x^*M \otimes \mathbb{C}$.

Definition 1.3.19. The *cotangent bundle (resp. holomorphic cotangent bundle)* T^*M (resp. H^*M) of a C^r manifold (resp. complex manifold) M is the vector bundle given by the above bijections Φ_U via the vector bundle chart lemma. The *tangent bundle (resp. holomorphic tangent bundle)* TM (resp. HM) of M is the dual vector bundle of T^*M (resp. H^*M).

Remark 1.3.20. The tangent or cotangent bundle of a differentiable manifold of class C^r is a differentiable manifold of class C^{r-1}.

Definition 1.3.21. A *differential n-form* (resp. *holomorphic n-form*) over $U \subseteq M$ is a section $\omega \in \Gamma(\bigwedge^n T^*M, U)$ (resp. $\Gamma(\bigwedge^n H^*M, U)$). A *vector field* (resp. *holomorphic vector field*) over an open subset $U \subseteq M$ is a section $X \in \Gamma(TM, U)$ (resp. $\Gamma(HM, U)$).

From now on, we will only work with the C^∞ case.

Remark 1.3.22. Given $f \in C^\infty(U)$ (resp. $\mathcal{H}(U)$), its *differential* $\mathrm{d}f \in \Gamma(T^*M, U)$ (resp. $\Gamma(H^*M, U)$) is the section $x \mapsto \mathrm{d}f_x$, which is C^∞ (resp. holomoprhic) by Corollary 1.3.17. Given some $X \in \Gamma(TM, U)$ (resp. $\Gamma(HM, U)$), denote $X(\mathrm{d}f) \in C^\infty(U)$ (resp. $\mathcal{H}(U)$) by Xf. We see X is a linear operator on $C^\infty(U)$ (resp. $\mathcal{H}(U)$) satisfying the product rule, $X(fg) = fXg + gXf$.

Remark 1.3.23. Given a C^r-chart (holomorphic chart) on $U \subseteq M$ with associated local C^r-coordinates $x^1, \ldots, x^n$ (resp. local holomorphic coordinates $z^1, \ldots, z^n$), the associated *coordinate frames* of T^*M and TM (resp. H^*M and HM) over U are the local frame $\mathrm{d}x^1, \ldots \mathrm{d}x^n$ (resp. $\mathrm{d}z^1, \ldots \mathrm{d}z^n$) and its dual frame, denoted $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ (resp. $\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}$), respectively.

Remark 1.3.24. Let M be a complex manifold with local holomorphic coordinates $z^1, \ldots, z^n$ and local C^∞-coordinates $x^1, \ldots, x^n, y^1, \ldots, y^n$ which are identified via $z^l = x^l + iy^l$. We have the holomorphic coordinate frame $\mathrm{d}z^1, \ldots, \mathrm{d}z^n$ for H^*M, where $\mathrm{d}z^l = \mathrm{d}x^l + i\mathrm{d}y^l$. We may extend $\mathrm{d}z^1, \ldots, \mathrm{d}z^n$ to a frame $\mathrm{d}z^1, \ldots, \mathrm{d}z^n, \mathrm{d}\bar{z}^1, \ldots, \mathrm{d}\bar{z}^n$ for $T^*M \otimes \mathbb{C}$, where $\mathrm{d}\bar{z}^l = \mathrm{d}x^l - i\mathrm{d}y^l$ and $\mathrm{d}\bar{z}^l$ is called the *complex conjugate* of $\mathrm{d}z^l$. More generally, given $\omega = \sigma + i\tau \in T_x^*M \otimes \mathbb{C}$ for $\sigma, \tau \in T_x^*M$, its complex conjugate is $\bar{\omega} = \sigma - i\tau$. The dual of the frame $\mathrm{d}\bar{z}^1, \ldots \mathrm{d}\bar{z}^n$ is denoted $\frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n}$. We have $\frac{\partial}{\partial z^l} = \frac{1}{2}\left(\frac{\partial}{\partial x^l} - i\frac{\partial}{\partial y^l}\right)$ and $\frac{\partial}{\partial \bar{z}^l} = \frac{1}{2}\left(\frac{\partial}{\partial x^l} + i\frac{\partial}{\partial y^l}\right)$ by direct computation, and consequently $f \in \mathcal{H}(U)$ if and only if $\frac{\partial}{\partial \bar{z}^l} f = 0$ for $f \in C^\infty(U) \otimes \mathbb{C}$ by Definition 1.1.6.

Remark 1.3.25. We write $T_{\mathbb{C}}^*M = T^*M \otimes \mathbb{C}$ and $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$.

Proposition 1.3.26. *The set of linear operators X on $C^\infty(U)$ (resp. $\mathcal{H}(U)$, $C^\infty(U) \otimes \mathbb{C}$) such that $X(fg) = gXf + fXg$ may be identified with $\Gamma(T^*M, U)$ (resp. $\Gamma(H^*M, U)$, $\Gamma(T_{\mathbb{C}}^*M, U)$).*

Proof. Given a linear operator X on $C^\infty(U)$ (resp. $\mathcal{H}(U)$) satisfying the product rule. Suppose $f \in C^\infty(U)$ (resp. $\mathcal{H}(U)$) such that $\mathrm{d}f_x = 0$, then $\frac{\partial f}{\partial x^l}|_x = 0$, so $Xf|_x = 0$ by Taylor's theorem (resp. Osgood's lemma) and the product rule. Thus X passes through the quotient to a smooth section in $\Gamma(TM, U)$ (resp. $\Gamma(HM, U)$). The statement follows from Remark 1.3.22. $\square$

Remark 1.3.27. The *Lie bracket* on $\Gamma(TM, U)$ (resp. $\Gamma(HM, U)$, $\Gamma(T^*_{\mathbb{C}}M)$) is defined $[X, Y] = XY - YX$. One may check $[X, Y]$ is linear and satisfies the product rule, and that the *Jacobi identity*, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ holds. In local coordinates $X = X^i \frac{\partial}{\partial x_i}$, $Y = Y^i \frac{\partial}{\partial x_i}$, we have

$$\begin{aligned}
[X, Y]f =& \left(X^j \frac{\partial}{\partial x^j}\right)\left(Y^i \frac{\partial}{\partial x^i} f\right) - \left(Y^j \frac{\partial}{\partial x^j}\right)\left(X^i \frac{\partial}{\partial x^i} f\right) \\
=& X^j \left(\frac{\partial}{\partial x^j} Y^i\right) \frac{\partial}{\partial x^i} f + X^j \left(\frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f\right) Y^i \\
& - Y^j \left(\frac{\partial}{\partial x^j} X^i\right) \frac{\partial}{\partial x^i} f - Y^j \left(\frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f\right) X^i \\
=& \left(X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial}{\partial x^j} X^i\right) \frac{\partial}{\partial x^i} f\,.
\end{aligned}$$

Remark 1.3.28. Given a C^1 curve (resp. holomorphic curve) γ passing x at t_0 (resp. z_0), the operator $\frac{\mathrm{d}(-)\circ\gamma}{\mathrm{d}t}|_{t=t_0}$ (resp. $\frac{\mathrm{d}(-)\circ\gamma}{\mathrm{d}z}|_{z=z_0}$) satisfies the product rule, so it is an element of T_xM (resp. H_xM). We denote it as $\gamma'(t_0)$ (resp. $\gamma'(z_0)$).

Remark 1.3.29. We write $\Omega^p(M) = \Gamma(\bigwedge^p T^*M)$. Define $\mathrm{d} : \Omega^1(M) \to \Omega^2(M)$ via $\mathrm{d}(\mathrm{d}f) = 0$ where $\mathrm{d}f$ is the differential of f for $f \in C^\infty(M)$, and extending to $\Omega^p(M)$ by following $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta$ for $\alpha \in \Omega^p(M)$. This is called the *exterior derivative* and it is the coboundary map of the cochain complex $\Omega^\bullet(M)$, the pth cohomology of which is the *pth de Rham cohomology* $H^p_{dR}(M)$ on M. In particular, d on $\Omega^1(M)$ extends by $\mathbb{C}$-linearity to $\mathrm{d} : \Omega^1_{\mathbb{C}}(M) \to \Omega^2_{\mathbb{C}}(M)$, which extends to coboundary maps of $\Omega^\bullet_{\mathbb{C}}(M)$. For a $\omega \in \Omega^k_{\mathbb{C}}(M)$, we have explicitly

$$\begin{aligned}
\mathrm{d}\omega(X_0, \dots, X_k) =& \sum_i X_i \omega(X_0, \dots, \widehat{X_i}, \dots, X_k) \\
& - \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots \widehat{X_j}, \dots, X_k)\,.
\end{aligned}$$

The Poincaré lemma states that $H^p_{dR}(U) = 0$ for any star-shaped $U \subseteq \mathbb{R}^n$ and $p \geq 1$. This implies that every closed real p-form is locally exact, meaning if $\mathrm{d}\omega = 0$ then there is a section τ over some neighbourhood such that $\mathrm{d}\tau = \omega$. A proof is found in [1, Theorem 17.14].

Definition 1.3.30. Given a C^∞ map $f : M \to N$, for differentiable manifolds M, N of class C^∞, the *pullback* is the $\mathbb{R}$-vector bundle morphism $f^* : f^*(T^*N) \to T^*M$ define by $f^*(\mathrm{d}g_{f(x)}) = \mathrm{d}(g \circ f)_x$ on each $T^*_{f(x)}N$ for $g \in C^\infty(U)$, $f(x) \in U \subseteq N$. The *pushforward* is the $\mathbb{R}$-vector bundle morphism $f_* : TM \to f^*(TN)$ defined by $f_*(X_p)g = X_p(g \circ f)$ on T_pM for $g \in C^\infty(U)$, $f(p) \in U \subseteq N$. The map f is an *immersion* if f_* has constant nullity 0, and a *submersion* if f_* has constant rank $\dim N$.

Definition 1.3.31. Let M, N be differentiable manifolds of class C^∞, then N is an *immersed submanifold* of M if there is an immersion $\iota : N \to M$, and an *embedded submanifold* (we will just call it a submanifold) of M if ι is additionally a homeomorphism onto its image. The *codimension* of N in M is $\dim M - \dim N$.

Definition 1.3.32. A subbundle Δ of TM (resp. $T_{\mathbb{C}}M$) is *involutive* if it is closed under Lie brackets. A subbundle Δ of TM is *integrable* if for each $x \in M$ there is a submanifold $N \ni x$ of M such that $\Delta_p = T_pN$ for all $p \in N$, where N is called an *integral manifold* of Δ.

Remark 1.3.33 (Frobenius theorem)**.** A subbundle Δ of TM is involutive if and only if it is integrable. We will also say an *integrable* subbundle of $T_{\mathbb{C}}M$ to mean an involutive subbundle of $T_{\mathbb{C}}M$. A proof is found in [1, Theorem 19.12].

Definition 1.3.34. A *connection* on a $\mathbb{K}$-vector bundle E over a differentiable manifold M of class C^∞ is a $\mathbb{K}$-linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the *Leibniz rule*, $\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s$ for $f \in C^r(M)$, $s \in \Gamma(E)$, for a finite field extension $\mathbb{K}$ of $\mathbb{R}$, or just $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Remark 1.3.35. A connection ∇ is determined locally, meaning $\nabla_X s|_p$ depends only on the value of $s \in \Gamma(E)$ on a neighbourhood of p, and we may define the restriction $\nabla : \Gamma(E, U) \to \Gamma(T^*M \otimes E, U)$ to some open subset $U \subseteq M$.

Remark 1.3.36. Given $\mathbb{K}$-vector bundles E, F with connections ∇, we have an associated connection ∇ on E^*, $E \oplus F$, $E \oplus F$, $\bigwedge^n E$ and $\mathrm{Sym}^n E$ defined respectively by:

1. $\nabla_X(\omega)(s) = X(\omega(s)) - \omega(\nabla_X s)$ for $\omega \in \Gamma(E^*)$, $s \in \Gamma(E)$, $X \in \Gamma(TM)$,
2. $\nabla(s \oplus t) = \nabla(s) \oplus \nabla(t)$ for $s \in \Gamma(E)$ and $t \in \Gamma(F)$,
3. $\nabla(s \otimes t) = \nabla(s) \otimes t + s \otimes \nabla(t)$ for $s \in \Gamma(E)$ and $t \in \Gamma(F)$,
4. $\nabla(s_1 \wedge \cdots \wedge s_n) = \sum_{i=1}^n s_1 \wedge \cdots \wedge \nabla(s_i) \wedge \cdots \wedge s_n$ for $s_1, \ldots, s_n \in \Gamma(E)$,
5. $\nabla(s_1 \odot \cdots \odot s_n) = \sum_{i=1}^n s_1 \odot \cdots \odot \nabla(s_i) \odot \cdots \odot s_n$ for $s_1, \ldots, s_n \in \Gamma(E)$.

If $k \mapsto \bar{k}$ is an involution on $\mathbb{K}$ that is a field automorphism (or just conjugation on $\mathbb{K} = \mathbb{C}$), we may also define ∇ on $\bar{E}$ by $\nabla_X \bar{s} = \overline{\nabla_X s}$. One may check that 3 to 5 above is well-defined.

Definition 1.3.37. A connection ∇ on a $\mathbb{K}$-vector bundle E equipped with a Riemannian metric (resp. Hermitian metric) h is *h-compatible* if $\nabla h = 0$.

Definition 1.3.38. A connection ∇ on TM is *torsion-free* if $\nabla_X Y - \nabla_Y X = [X, Y]$.

1.4 Riemannian manifolds

Everything in this section can be found in [2].

Definition 1.4.1. A *Riemannian manifold* is a differentiable manifold M of class C^∞ equipped with a Riemannian metric g on TM.

Remark 1.4.2 (Fundamental theorem of Riemannian geometry)**.** There exists a unique *Levi-Civita connection* ∇ on a Riemannian manifold (M, g) that is torsion-free and g-compatible.

Definition 1.4.3. The *curvature tensor* $R \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes TM)$ is given by $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, and the associated tensor $R \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes T^*M)$ is given by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Remark 1.4.4. The curvature tensor R has the following symmetries and identities:

1. (skew symmetry) $R(X, Y) = -R(Y, X)$,

2. (skew symmetry) $R(X,Y,Z,W) = -R(X,Y,W,Z)$,

3. (interchange symmetry) $R(X,Y,Z,W) = R(Z,W,X,Y)$,

4. (first Bianchi identity) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$,

5. (second Bianchi identity) $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$.

Remark 1.4.5. Given a $\mathbb{K}$-vector space V, there is a canonical identification $V^* \otimes V = \mathrm{End}(V)$. In particular, we may define $\mathrm{tr} : V^* \otimes V \to \mathbb{K}$.

Definition 1.4.6. The *Ricci curvature tensor* $\mathrm{Ric} \in \Gamma(T^*M \otimes T^*M)$ is given by $\mathrm{Ric}(X,Y) = \mathrm{tr}\, R(-,X)Y$ and the associated tensor $\mathrm{Ric} \in \Gamma(T^*M \otimes TM)$ is given by $g(\mathrm{Ric}(X), -) = \mathrm{Ric}(X,-)$. The *scalar curvature* $R_s \in C^\infty(M)$ is given by $R_s(x) = \mathrm{tr}\,\mathrm{Ric}_x(-)$.

Remark 1.4.7. Given a local g-orthonormal frame $X_1, \dots, X_n \in \Gamma(TM, U)$, we have $\mathrm{Ric}(X,Y) = \sum_{i=1}^n R(X_i, X, Y, X_i)$, which by the symmetries of R shows that Ric is symmetric. Also, $R_s(x) = \sum_{i=1}^n \mathrm{Ric}(X_i, X_i)$.

Definition 1.4.8. We write $\mathrm{Ric} \geq k$ for $k \in \mathbb{R}$ if all eigenvalues λ of $\mathrm{Ric}(-)$ satisfies $\lambda \geq k$. A Riemannian manifold (M,g) is *Einstein* with *Einstein constant* $k \in \mathbb{R}$ if $\mathrm{Ric}(X) = kX$ for all $X \in \Gamma(TM)$. This means $\mathrm{Ric}(X,Y) = kg(X,Y)$ for all $X \in \Gamma(TM)$ by Definition 1.4.6.

Definition 1.4.9. The *sectional curvature* K of M is a real function on the fibre bundle $\mathrm{Gr}(2,M)$ of 2-dimensional subspaces of T_xM given by $K(\sigma) = R(X,Y,Y,X)$ for any orthonormal basis X, Y of $\sigma \subseteq T_xM$.

Definition 1.4.10. Let (M,g) be a Riemannian manifold. A *Riemannian submanifold* (N,g) of (M,g) is a submanifold N of M equipped with the induced Riemannian metric from g. The *normal bundle* $T^\perp N$ over N is defined by $T_x^\perp N = (T_xN)^\perp$ where $\perp$ indicates the g-orthogonal complement in TM. The *second fundamental form* of N is $\mathrm{I\!I} \in \Gamma(T^*N \otimes T^*N \otimes T^\perp N)$ given by $\mathrm{I\!I}(X,Y) = (\nabla_X Y)^\perp$, where $\perp$ indicates the orthogonal projection onto $T^\perp N$.

Remark 1.4.11 (Gauss-Codazzi formula)**.** Let $N \subseteq M$ be a submanifold of (M,g), R_N be the Riemann curvature tensor on N, and $\mathrm{I\!I}$ be the second fundamental form of N. We have $\mathrm{I\!I}(X,Y) - \mathrm{I\!I}(Y,X) = [X,Y]^\perp = 0$ since $[X,Y] \in \Gamma(TN)$ for $X, Y \in \Gamma(TN)$. Hence $\mathrm{I\!I}$ is symmetric. Moreover, we have the Gauss-Codazzi formula:

$$R_N(X,Y,Z,W) = R(X,Y,Z,W) + g(\mathrm{I\!I}(X,W), \mathrm{I\!I}(Y,Z)) - g(\mathrm{I\!I}(X,Z), \mathrm{I\!I}(Y,W)).$$

Definition 1.4.12. A C^∞ curve $\gamma : I \to M$ on a Riemannian manifold is a geodesic if $\nabla_{\gamma'(t)}\gamma'(t) = 0$ for all $t \in I$.

Remark 1.4.13. The tangent vector $\nabla_{\gamma'(t)}\gamma'(t)$ is defined by extending $\gamma'(s)$ for $s \in (t-\varepsilon, t+\varepsilon)$, for some $\varepsilon > 0$ to a C^∞ section over a neighbourhood of $\gamma(t)$. The result is indeed independent of the choice of extension.

Remark 1.4.14. Given a $v \in T_xM$, there is a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x$ and $\gamma'(0) = v$, for $\varepsilon > 0$ small enough.

Definition 1.4.15. The *exponential map* $\exp_x : U \subseteq T_xM \to M$ at $x \in M$ is given by $\exp_x(v) = \gamma_v(1)$, where γ_v is the unique geodesic with $\gamma_v'(0) = v$, and U is a neighbourhood of 0 where $\gamma_v(1)$ is defined for all $v \in U$.

Remark 1.4.16. For small enough U, $\exp_x$ is a C^∞-diffeomoprhism. Given a basis $v_1, \ldots, v_n$ of T_xM with dual basis $v^1, \ldots, v^n$, we have the chart $\exp_x^{-1} : \exp_x(U) \mapsto U \subseteq T_xM \cong \mathbb{R}^n$, and $v^i = v^i \circ \exp_x^{-1}$ are the associated local coordinates. Local coordinates associated with charts arising this way are called *normal coordinates.*

Proposition 1.4.17 (First and second variation of length)**.** *Suppose V, W are submanifolds of a complete Riemannian manifold (M, g), and γ a shortest geodesic joining V and W, where X is a unit parallel field, and $c_\alpha(t)$ a smooth variation of $c_0(t) = \gamma(t)$ and $\frac{\partial}{\partial\alpha}c_\alpha(t)|_{\alpha=0} = X_{\gamma(t)}$. Denoting $T_{c_\alpha(t)} = \frac{\partial}{\partial t}c_\alpha(t)$, and let $L(\alpha) = \int_0^l \sqrt{g(T_{c_\alpha(t)}, T_{c_\alpha(t)})}\mathrm{d}t$ be the arc length of γ, then the first and second variations of length for γ with variational field X are*

$$L_X'(0) = \frac{\partial}{\partial\alpha}L(\alpha)|_{\alpha=0} = 0\,,$$

$$L_X''(0) = \frac{\partial^2}{\partial\alpha^2}L(\alpha)|_{\alpha=0} = g_q(\nabla_X X, T) - g_p(\nabla_X X, T) - \int_0^l R(T, X, X, T)\mathrm{d}t\,.$$

If $c_\alpha(t_0)$ is a geodesic for constant t_0, then $L_X''(0) = -\int_0^l R(T, X, X, T)\mathrm{d}t$.

1.5 Kähler metrics

The goal in this section is be to give a basic understanding of Kähler manifolds.

Almost complex structures

A Kähler manifold has a Hermitian structure and a complex structure. We start by defining a complex structure.

Definition 1.5.1. Given a differentiable manifold M of class C^∞, an *almost complex structure* on M is a section $J \in \Gamma(T^*M \otimes TM)$ such that $J^2 = -1$ as a section of $\mathrm{End}(TM)$. The pair (M, J) is called an *almost complex manifold.*

Remark 1.5.2. An almost complex structure has a unique extension $J \in \Gamma(T_\mathbb{C}^*M \otimes T_\mathbb{C}M)$. Since $J^2 = -1$, the only possible eigenvalues of J are $i, -i$, JX is linearly independent to X for $X \in T_pM$, and J is a non-singular endomorphism at each point.

Definition 1.5.3. Let $T^{0,1}M$ be the subbundle of i-eigenspaces of J and $T^{1,0}M$ be the subbundle of $-i$-eigenspace of J.

Proposition 1.5.4. *We have $T_\mathbb{C}M = T^{0,1}M \oplus T^{1,0}M$, and $T^{0,1}M = \{X + iJX : X \in TM\}$ and $T^{1,0}M = \{X - iJX : X \in TM\}$, so $T^{0,1}M$ and $T^{1,0}M$ are indeed subundles.*

Proof. Let $n = \dim M$. Since $J^2 = -1$, J does not have real eigenvalues, so X, JX are linearly independent for every $X \in TM$. Also by $J^2 = -1$, if $X_1, JX_1, \ldots, X_r, JX_r, Y$ are linearly independent, then so are $X_1, JX_1, \ldots, X_r, JX_r, JY$, thus we can find by induction a basis for T_xM of the form $X_1, JX_1, \ldots, X_n, JX_n$. Clearly $X + iJX$ and $X - iJX$ is a $-i$-eigenvector and i-eigenvector respectively for any $X \in TM$, and $X_1 + iJX_1, X_1 - iJX_1, \ldots, X_n + iJX_n, X_n - iJX_n$ is a $\mathbb{C}$-basis for $(T_\mathbb{C}M)_x$. Hence we get the desired decomposition of $T_\mathbb{C}M$ and explicit descriptions of $T^{0,1}M$ and $T^{1,0}M$. Each of these has $\mathbb{R}$-rank n, so they are indeed subbundles. $\square$

Remark 1.5.5. An almost complex manifold (M, J) is always even dimensional since T_pM is the direct sum of invariant subspaces $\mathrm{span}_{\mathbb{R}}(X, JX)$ for $X \in T_pM$.

Remark 1.5.6. Similarly, we can consider J to be a section of $\mathrm{End}(T^*_{\mathbb{C}}M)$ where $J^2 = -1$, and do the above constructions with $T^*_{\mathbb{C}}M$ verbatim. Particularly, $J\omega = J(-, \omega) = \omega(J(-)) = \omega \circ J$, where the last two J are considered to be a section of $\mathrm{End}(T_{\mathbb{C}}M)$.

Definition 1.5.7. Let $\bigwedge^{1,0}M$ and $\bigwedge^{0,1}M$ be the subbundle of $T^*_{\mathbb{C}}M$ of $-i$-eigenspaces and i-eigenspaces of J respectively.

Proposition 1.5.8. *We have $T^*_{\mathbb{C}}M = \bigwedge^{1,0}M \oplus \bigwedge^{0,1}M$, $\bigwedge^{1,0}M = \{\omega - iJ\omega : \omega \in T^*M\}$ and $\bigwedge^{0,1}M = \{\omega + iJ\omega : \omega \in T^*M\}$, and $T^{0,1}M$ and $T^{1,0}M$ is the annihilator of $\bigwedge^{1,0}M$ and $\bigwedge^{0,1}M$ respectively.*

Proof. Suppose $X \in T^{0,1}_pM$ and $\omega \in \bigwedge^{1,0}_pM$, then $J(X, -) = iX(-)$ and $J(-, \omega) = -i(-)(\omega)$ as they are eigenvectors, so $iX(\omega) = J(X, \omega) = -iX(\omega)$, thus $X(\omega) = 0$. Similarly for $X \in T^{1,0}_pM$ and $\omega \in \bigwedge^{0,1}_pM$. $\square$

Definition 1.5.9. A *complex k-form* is a section of $\bigwedge^k T^*_{\mathbb{C}}M$ and a *(p, q)-form* is a section of $\bigwedge^{p,q}M$, where $\bigwedge^{p,0}M = \bigwedge^p(\bigwedge^{1,0}M)$, $\bigwedge^{0,q}M = \bigwedge^q(\bigwedge^{0,1}M)$, and $\bigwedge^{p,q}M = \bigwedge^{p,0}M \otimes \bigwedge^{0,q}M$.

Remark 1.5.10. We have $\bigwedge^k T^*_{\mathbb{C}}M = \bigoplus_{p+q=k} \bigwedge^{p,q}M$, where $\bigwedge^{p,q}_xM$ is identified with the space generated by pure tensors $\sigma \wedge \tau$ for $\sigma \in \bigwedge^{p,0}_xM$ and $\tau_x \in \bigwedge^{0,q}M$. Write $\bigwedge T^*_{\mathbb{C}}M = \bigoplus_{k=0}^n \bigwedge^k T^*_{\mathbb{C}}M$, and $\bigwedge^{0,0}M = \mathbb{R}_M \otimes \mathbb{C}$, where $\mathbb{R}_M$ is the trivial vector bundle.

Proposition 1.5.11. *A complex k-form ω is a (p, q)-form if and only if $\omega(X_1, \ldots, X_k) = 0$ whenever there are $p+1$ sections of $T^{0,1}M$ or $q+1$ sections of $T^{1,0}M$ in $X_1, \ldots, X_k$.*

Proof. We prove the forward direction on pure tensors $\sigma \otimes \tau$ for $\sigma \in \bigwedge^{p,0}_xM$ and $\tau \in \bigwedge^{0,q}_xM$. If $X_1, \ldots, X_k \in (T_{\mathbb{C}}M)_x$ has $p+1$ elements of $T^{0,1}_xM$ or $q+1$ elements of $T^{1,0}_xM$, then $\sigma \otimes \tau(X_1, \ldots, X_k) = \sigma(X_1, \ldots, X_p)\tau(X_{p+1}, \ldots, X_k)$ has at least one $X_i \in T^{0,1}_xM$ in the arguments of σ or at least one $X_i \in T^{1,0}_xM$ in the arguments of τ, so $\sigma \otimes \tau(X_1, \ldots, X_k) = 0$. For the backwards direction, consider a complex k-form $\omega = \omega_1 \wedge \cdots \wedge \omega_k$ that is a pure tensor satisfying the condition, such that each ω_i is either in $\bigwedge^{1,0}_xM$ or $\bigwedge^{0,1}_xM$, then the condition implies there are exactly p-many $\omega_i \in \bigwedge^{1,0}_xM$ and q-many $\omega_i \in \bigwedge^{1,0}_xM$ in $\omega_1, \ldots, \omega_k$. $\square$

Remark 1.5.12. For $\sigma \in \bigwedge^{p,q}_xM$ and $\tau \in \bigwedge^{r,s}_xM$, we have $\sigma \wedge \tau \in \bigwedge^{p+r,q+s}_xM$, where $\bigwedge^{a,b}M = 0$ if $a + b > \dim M = 2n$ or $\max\{a, b\} > n$. Given a pure complex k-form $\omega = \omega_1 \wedge \cdots \wedge \omega_k$, its *complex conjugate* is $\bar{\omega} = \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k$, and we may extend this definition by linearity to all complex k-forms. In particular, $\bar{\omega} \in \bigwedge^{q,p}M$ for $\omega \in \bigwedge^{p,q}M$, and d commutes with complex conjugation.

Proposition 1.5.13. *We have $\mathrm{d}(\bigwedge^{p,q}M) \subseteq \bigwedge^{p+2,q-1}M \oplus \bigwedge^{p+1,q}M \oplus \bigwedge^{p,q+1}M \oplus \bigwedge^{p-1,q+2}M$ for an almost complex manifold.*

Proof. By Remark 1.5.10, $\mathrm{d}(\bigwedge^{0,0}M) \subseteq \bigwedge^{1,0}M \oplus \bigwedge^{0,0}M$ and $\mathrm{d}(\bigwedge^{1,0}M), \mathrm{d}(\bigwedge^{0,1}M) \subseteq \bigwedge^{0,2}M \oplus \bigwedge^{1,1}M \oplus \bigwedge^{0,2}M$. The statement follows from $\mathrm{d}(\sigma \wedge \tau) = \mathrm{d}\sigma \wedge \tau - \sigma \wedge \mathrm{d}\tau$ for complex 1-form σ and complex k-form τ, and induction. $\square$

Definition 1.5.14. The linear operators $\partial : \bigwedge^{p,q} M \to \bigwedge^{p+1,q} M$ and $\bar{\partial} : \bigwedge^{p,q} M \to \bigwedge^{p,q+1} M$ are defined as $\partial = \pi^{p+1,q} \circ \mathrm{d}$ and $\bar{\partial} = \pi^{p,q+1} \circ \mathrm{d}$ where $\pi^{p,q} : \bigwedge T^*_{\mathbb{C}} M \to \bigwedge^{p,q} M$ is the projection map, for an almost complex manifold M. One can check that $\partial, \bar{\partial}$ satisfy Leibniz's rule.

Definition 1.5.15. The *Nijenhuis tensor* $N^J \in \Gamma(T^*M \otimes T^*M \otimes TM)$ is given by $N^J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$ for an almost complex manifold (M,J).

Proposition 1.5.16. *The Nijenhuis tensor N^J is a tensor.*

Proof. In local coordinates, by Remark 1.3.27, we have $[X,Y] = \partial_X Y - \partial_Y X$, where ∂_X is given by component-wise differentiation, for example, $\partial_X Y = X^j \frac{\partial}{\partial x^j}(Y^i)\frac{\partial}{\partial x^i}$. We may also apply ∂_X component-wise to J, which is a matrix of functions under the local coordinates, to get another matrix of functions. Then

$$\begin{aligned}
[X,Y] =& \partial_X Y - \partial_Y X\,,\\
[JX,JY] =& \partial_{JX}(JY) - \partial_{JY}(JX)\\
=& (\partial_{JX} J)Y + J\partial_{JX}(Y) - (\partial_{JY} J)X - J\partial_{JY}(X)\,,\\
J[JX,Y] =& J\partial_{JX}(Y) - J\partial_Y(JX)\\
=& J\partial_{JX}(Y) - (\partial_Y J)(JX) + \partial_Y X\,,\\
J[X,JY] =& (\partial_X J)(JY) - \partial_X Y - J\partial_{JY}(X)\,,
\end{aligned}$$

thus

$$N^J(X,Y) = (\partial_{JY} J)X - (\partial_{JX} J)Y + (\partial_X J)(JY) - (\partial_Y J)(JX)\,,$$

which satisfies $N^J(fX, gY) = fgN^J(X,Y)$ for $X, Y \in \Gamma(TM)$ and $f, g \in C^\infty(M)$. $\square$

Remark 1.5.17. Given a complex manifold M, and a holomorphic chart (U,ϕ) with associated local coordinates $z^1, \dots, z^n$, and local C^∞-coordinates $x^1, \dots, x^n, y^1, \dots, y^n$ where $z^l = x^l + iy^l$, we may define a natural complex structure J by $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$ and $J(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}$. This is the same as $J = \varphi_*^{-1} \circ j_n \circ \varphi_*$, so given another holomorphic chart (V,ψ), we have $\varphi_*^{-1} \circ j_n \circ \varphi_* = \psi_*^{-1} \circ \psi_* \circ \varphi_*^{-1} \circ j_n \circ \varphi_* \circ \psi_*^{-1} \circ \psi_* = \psi_*^{-1} \circ j_n \circ \psi_* \circ \varphi_*^{-1} \circ \varphi_* \circ \psi_*^{-1} \circ \psi_* = \psi_*^{-1} \circ j_n \circ \psi_*$ by Definition 1.1.6 since $\psi_* \circ \varphi_*^{-1} = (\psi \circ \varphi^{-1})_*$ is holomorphic, thus J is well-defined. Then $\mathrm{d}z^1, \dots, \mathrm{d}z^n$ and $\mathrm{d}\bar{z}^1, \dots, \mathrm{d}\bar{z}^n$ is a local frame for $\bigwedge^{1,0} M$ and $\bigwedge^{0,1} M$ respectively. From Remark 1.3.24, a function $f \in C^\infty(U) \otimes \mathbb{C}$ is holomorphic if and only if $Zf = 0$ for all $Z \in T^{0,1}M$ if and only if $\mathrm{d}f \in \bigwedge^{1,0} M$. Given two complex manifolds (M,J) and (N,J'), a C^∞-map $f : M \to N$ is holomorphic if and only if $f_* \circ J = J' \circ f_*$, where J, J' are viewed as operators on the tangent spaces. A *complex structure* is an almost complex structure arising from a holomorphic structure this way. We see that $H^*M = \bigwedge^{1,0} M$ under the identification in Proposition 1.3.14, by Corollary 1.3.17 and Remark 1.3.24.

Proposition 1.5.18. *Given an almost complex manifold (M,J), the following are equivalent:*

1. *J is a complex structure;*
2. *$T^{0,1}M$ is involutive;*

3. $\mathrm{d} = \partial + \bar\partial$;

4. $\bar\partial^2 = 0$;

5. $N^J = 0$.

Proof. ($1 \implies 2$) Given $Z, W \in T_x^{0,1}M$, written in local coordinates $Z = Z^l \frac{\partial}{\partial \bar z^l}$ and $W = W^l \frac{\partial}{\partial \bar z^l}$, then $[Z, W] \in T_x^{0,1}M$ by the local coordinate expression in Remark 1.3.27.

($2 \implies 1$) This is a deep theorem of Newlander and Nirenberg. A proof may be found in [5, Chapter 2].

($2 \iff 3$) Given $\omega \in \Gamma(\bigwedge^{1,0}M)$, and $Z, W \in \Gamma(T^{0,1}M)$, $\mathrm{d}\omega(Z, W) = Z\omega(W) + W\omega(Z) - \omega([Z, W]) = 0$, thus $\mathrm{d}\omega \in \Gamma(\bigwedge^{1,1}M \oplus \bigwedge^{0,2}M)$, so $\mathrm{d} = \partial + \bar\partial$ on $\Gamma(\bigwedge^{1,0}M)$, and on $\Gamma(\bigwedge^{0,1}M)$ as well since d commutes with complex conjugation. The rest follows by induction as in Proposition 1.5.13. On the other hand, if $\mathrm{d} = \partial + \bar\partial$, then $\omega([Z, W]) = 0$, so $T^{0,1}M$ is integrable.

($3 \implies 4$) Since $\mathrm{d}^2 = \partial^2 + \partial\bar\partial + \bar\partial\partial + \bar\partial^2$, where each term maps to a different space, so $\partial^2 = 0$, $\bar\partial^2 = 0$, and $\partial\bar\partial + \bar\partial\partial = 0$.

($4 \implies 3$) Let e^i be a local frame for $\bigwedge^{1,0}M$. Given a function F on a neighbourhood of x, write $\mathrm{d}F = f_i e^i + g_i \bar e^i$, so $\partial F = f_i e^i$. Then $0 = \bar\partial^2 F = \pi^{0,2}\mathrm{d}\bar\partial F = \pi^{0,2}\,\mathrm{d}(\mathrm{d} - \partial)\,F = -\pi^{0,2}\mathrm{d}\partial F = -\pi^{0,2}\,\mathrm{d}(f_i e^i)$, so the $(0,2)$-part of $\mathrm{d}(f_i e_i)$ vanishes. Then $\mathrm{d}\omega_x \in \bigwedge_x^{1,1}M \oplus \bigwedge_x^{0,2}M$ for all $(1,0)$-forms ω.

($2 \iff 5$) Let $X, Y \in TM$, and $Z = [X + iJX, Y + iJY] = [X, Y] + i[X, JY] + i[JX, Y] - [JX, JY]$. Then

$$
\begin{aligned}
&Z - iJZ \\
=&[X, Y] + i[X, JY] + i[JX, Y] - [JX, JY] - iJ[X, Y] + J[X, JY] + J[JX, Y] + iJ[JX, JY] \\
=&N^J(X, Y) - iJN^J(X, Y)\,.
\end{aligned}
$$

So $Z \in T^{0,1}M$ if and only if $N^J = 0$. □

Complex structures

We list some facts about complex structures. In this subsection, M will be a complex manifold.

Proposition 1.5.19. *A smooth complex function $f : U \to \mathbb{C}$ defined locally on M is holomorphic if and only if $Zf = 0$ for every Z of type $(0,1)$, if and only if $\mathrm{d}f$ is type $(1,0)$.*

Proof. The second equivalence is clear. For the first equivalence, using a local holomorphic coordinate, f is holomorphic if and only if $(JX)f = iXf$ for all X, if and only if $i(X + iJX)f = 0$ for all X. □

Definition 1.5.20. A vector field $Z \in \Gamma(T^{0,1}M)$ is *holomorphic* of $Z(f) \in \mathcal{H}(U)$ for every $f \in \mathcal{H}(U)$, $U \subseteq M$ an open subset. A $(p,0)$-form ω is *holomorphic* if $\bar\partial\omega = 0$. A real vector field $X \in \Gamma(TM)$ is *holomorphic* if $X - iJX$ is holomorphic.

Proposition 1.5.21. *A real vector field X is holomorphic if and only if $\mathcal{L}_X J = 0$, if and only if the flow of X are holomorphic transformations.*

Proof. The last two statements are logically equivalent. For the first equivalence, suppose X, Y are vector fields where X is real holomorphic, and f is a locally defined holomorphic function. Note Z is $(0,1)$ if and only if $Zf = 0$ for any locally defined holomorphic f. Thus $(X + iJX)f = 0$, so $(X - iJX)f = 2Xf$, then Xf is holomorphic. Hence $(Y + iJY)(Xf) = 0$ and $(Y + iJY)f = 0$, so $[Y + iJY, X]f = 0$. Since f was arbitrary, $[Y + iJY, X]$ is type $(0,1)$, so $[Y, X] + i[JY, X] = [Y + iJY, X] = [Y, X] + iJ[Y, X]$, thus $[JY, X] = J[Y, X]$. Then $(\mathcal{L}_X J)Y = \mathcal{L}_X(JY) - J\mathcal{L}_X Y = [JY, X] - J[Y, X] = 0$.

Similarly, suppose for all Y, $[JY, X] - J[Y, X] = (\mathcal{L}_X J)Y = 0$. Then $[Y + iJY, X]$ is $(0,1)$, thus for any holomorphic f, we have $(Y + iJY)(Xf) - X(Y + iJY)f = 0$. Since $(Y + iJX)$ is $(0,1)$, $X(Y + iJY)f = 0$, thus $(Y + iJY)(Xf) = 0$. So Xf is holomorphic since Y was arbitrary. Then $2Xf = (X + iJX)f + (X - iJX)f = (X - iJX)f$ is holomorphic, thus X is real holomorphic. □

Proposition 1.5.22 ($\bar\partial$-Poincaré lemma)**.** *For a $(0,1)$-form ω such that $\bar\partial\omega = 0$, there exists locally a function f such that $\bar\partial f = \omega$.*

A proof of the $\bar\partial$-Poincaré lemma is in [3, p. 25].

Proposition 1.5.23 (local $i\partial\bar\partial$-lemma)**.** *Let ω be a real 2-form of type $(1,1)$ on a complex manifold M, then ω is closed if and only if $\omega = i\partial\bar\partial u$ for some locally defined real function u.*

Proof. Suppose $\omega = i\partial\bar\partial u$, then $\mathrm{d}\omega = 0$ as $\mathrm{d}\big(\partial\bar\partial\big) = \partial^2\bar\partial + \partial\bar\partial^2 = 0$. Suppose ω is a closed real $(1,1)$-form. By the Poincaré lemma in Remark 1.3.29, there is a local real 1-form τ such that $\mathrm{d}\tau = \omega$. Decompose $\tau = \tau^{1,0} + \tau^{0,1}$, then $\overline{\tau^{0,1}} = \tau^{1,0}$ since τ is real. By $\omega = \mathrm{d}\tau = \partial\tau^{1,0} + \bar\partial\tau^{0,1} + (\bar\partial\tau^{1,0} + \partial\tau^{0,1})$, we have $\partial\tau^{1,0} = \partial\tau^{0,1} = 0$ as ω is type $(1,1)$. The $\bar\partial$-Poincaré lemma gives a local function f such that $\bar\partial f = \tau^{0,1}$, so $\tau^{1,0} = \partial\bar f$. Thus $\omega = \bar\partial\tau^{1,0} + \partial\tau^{0,1} = \bar\partial\partial\bar f + \partial\bar\partial f = i\partial\bar\partial(2\operatorname{Im} f)$. □

Non-degenerate 2-forms with complex structure

One way to look at Kähler manifolds is as a manifold with a closed non-degenerate 2-form with a compatible complex structure J.

Definition 1.5.24. Let ω be a non-degenerate 2-form over a C^∞-manifold M. An almost complex structure J on M is *ω-tame* if $\omega(X, J(X)) > 0$ for all $X \in TM$, and *ω-compatible* if it is ω-tame and $\omega(X, Y) = \omega(JX, JY)$.

Proposition 1.5.25. *Let ω be a J-compatible non-degenerate 2-form on an almost complex manifold (M, J), the Riemannian metric $g(-,-) = \omega(-, J(-))$ and its Levi-Civita connection ∇ satisfies*

$$(\nabla_X J)J + J(\nabla_X J) = 0\,,$$

$$g((\nabla_X J)Y, Z) + g(Y, (\nabla_X J)Z) = 0\,,$$

$$\mathrm{d}\omega(X, Y, Z) = g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y)\,.$$

If ω is closed, then

$$(\nabla_{JX} J) = -J(\nabla_X J)\,.$$

Proof. The first identity follows from differentiating $J^2 = -1$. By differentiating $g(JY, Z) + g(Y, JZ) = 0$, we get

$$\begin{aligned}
0 =& g(\nabla_X(JY), Z) + g(JY, \nabla_X Z) + g(\nabla_X Y, JZ) + g(Y, \nabla_X(JZ)) \\
=& g(\nabla_X(JY), Z) - g(Y, J(\nabla_X(Z))) - g(J(\nabla_X Y), Z) + g(Y, \nabla_X(JZ)) \\
=& g((\nabla_X J)Y, Z) + g(Y, (\nabla_X J)Z) .
\end{aligned}$$

With $\omega(X, Y) = -g(X, JY)$ and the second identity, we have

$$\begin{aligned}
d\omega(X, Y, Z) =& X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) - \omega([X, Y], Z) - \omega([Y, Z], X) + \omega([X, Z], Y) \\
=& X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\
& - \omega(\nabla_X Y - \nabla_Y X, Z) - \omega(\nabla_Y Z - \nabla_Z Y, X) + \omega(\nabla_X Z - \nabla_Z X, Y) \\
=& - Xg(Y, JZ) + Yg(X, JZ) - Zg(X, JY) \\
& + g(\nabla_X Y - \nabla_Y X, JZ) + g(\nabla_Y Z - \nabla_Z Y, JX) - g(\nabla_X Z - \nabla_Z X, JY) \\
=& - g(\nabla_X Y, JZ) + g(\nabla_Y X, JZ) - g(\nabla_Z X, JY) \\
& - g(Y, \nabla_X(JZ)) + g(X, \nabla_Y(JZ)) - g(X, \nabla_Z(JY)) \\
& + g(\nabla_X Y - \nabla_Y X, JZ) + g(\nabla_Y Z - \nabla_Z Y, JX) - g(\nabla_X Z - \nabla_Z X, JY) \\
=& - g(Y, \nabla_X(JZ)) + g(X, \nabla_Y(JZ)) - g(X, \nabla_Z(JY)) \\
& + g(\nabla_Y Z - \nabla_Z Y, JX) - g(\nabla_X Z, JY) \\
=& - g(Y, (\nabla_X J)Z) + g((\nabla_Y J)Z, X) - g(X, (\nabla_Z J)Y) \\
=& g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) .
\end{aligned}$$

For $X \in \Gamma(TM)$, define $\tau_X(Y, Z) = g((\nabla_X J)Y, Z)$. The second identity shows that τ_X is a 2-form. Using the first identity we get

$$\tau_X(Y, Z) + \tau_X(JY, JZ) = 0 .$$

By the third identity and $d\omega = 0$,

$$\tau_X(Y, Z) + \tau_Y(Z, X) + \tau_Z(X, Y) = 0 ,$$

then applying this to $\tau_X(Y, Z)$ and $\tau_X(JY, JZ)$ gives

$$2\tau_X(Y, Z) = \tau_X(Y, Z) - \tau_X(JY, JZ) = -\tau_Y(Z, X) - \tau_Z(X, Y) + \tau_{JY}(JZ, X) + \tau_{JZ}(X, JY) .$$

Replacing X, Y with JX, JY,

$$\begin{aligned}
2\tau_{JX}(JY, Z) =& - \tau_{JY}(Z, JX) - \tau_Z(JX, JY) - \tau_Y(JZ, JX) - \tau_{JZ}(JX, Y) \\
=& - \tau_{JY}(JZ, X) + \tau_Z(X, Y) + \tau_Y(Z, X) - \tau_{JZ}(X, JY) \\
=& - 2\tau_X(Y, Z) ,
\end{aligned}$$

which gives the last identity. □

Corollary 1.5.26. *Let ω be a J-compatible non-degenerate 2-form on an almost complex manifold (M, J), defining the Riemannian metric $g(-,-) = \omega(-, J(-))$ and its Levi-Civita connection ∇. The following are equivalent:*

1. $\nabla J = 0$;

2. *J is integrable and ω is closed.*

Proof. We use Proposition 1.5.25. Using $[X,Y] = \nabla_X Y - \nabla_Y X$, $\nabla_X(JY) = (\nabla_X J)Y + J\nabla_X Y$ and the first identity,

$$\begin{aligned} N^J(X,Y) =&[X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \\ =&\nabla_X Y - \nabla_Y X + J\nabla_{JX}Y - J\nabla_Y(JX) \\ &+ J\nabla_X(JY) - J\nabla_{JY}X - \nabla_{JX}(JY) + \nabla_{JY}(JX) \\ =&-(\nabla_{JX}J)Y + (\nabla_{JY}J)X - (\nabla_X J)JY + (\nabla_Y J)JX\,. \end{aligned}$$

Then $\nabla J = 0$ implies $N^J = 0$ and $\mathrm{d}\omega = 0$ by the above and the third identity respectively, since $\nabla_X J = 0$. Using the first three identities,

$$\begin{aligned} g(N^J(X,Y),Z) =&-g((\nabla_{JX}J)Y + (\nabla_{JY}J)X - (\nabla_X J)JY + (\nabla_Y J)JX, Z) \\ =&-g((\nabla_{JX}J)Y,Z) - g((\nabla_Y J)Z, JX) - g((\nabla_Z J)JX, Y) \\ &-g((\nabla_X J)JY, Z) - g((\nabla_{JY}J)Z, X) - g((\nabla_Z J)X, JY) - 2g(J(\nabla_Z J)X, Y) \\ =&-\mathrm{d}\omega(JX,Y,Z) - \mathrm{d}\omega(X,JY,Z) - 2g(J(\nabla_Z J)X,Y)\,, \end{aligned}$$

so if $N^J = 0$ and $\mathrm{d}\omega = 0$, then $\nabla J = 0$. ☐

Holomorphic vector bundles

We introduce Chern connections for the purpose of characterizing Kähler manifolds. Fix M to be a be a complex manifold.

Definition 1.5.27. Let $E \to M$ be a $\mathbb{C}$-vector bundle. An operator $\bar{\partial}_E : \bigwedge^{p,q} E \to \bigwedge^{p,q+1} E$, where $\bigwedge^{p,q} E = \bigwedge^{p,q} M \otimes E$, is a *pre-holomorphic structure* on E if it satisfies the Leibniz rule $\bar{\partial}_E(fs) = \bar{\partial}(f) \otimes s + f\bar{\partial}_E(s)$. If $\bar{\partial}_E$ is additionally a coboundary map, i.e. $\bar{\partial}_E^2 = 0$, it is a *holomorphic structure*.

Definition 1.5.28. A complex vector bundle $E \to M$ is *holomorphic* if there exists local trivializations Ψ_α, where $(\Psi_\alpha \circ \Psi_\beta^{-1})(x) = (x, g_{\alpha\beta}(x)v)$, such that the transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n,\mathbb{C})$ are holomorphic.

Remark 1.5.29. Given a pre-holomorphic vector bundle $(E, \bar{\partial}_E)$, a section $\sigma \in \Gamma(E)$ is called *holomorphic* if $\bar{\partial}_E \sigma = 0$.

Proposition 1.5.30. *A complex bundle is holomorphic if and only if it has a holomorphic structure. More specifically, there is a canonical holomorphic structure for every holomorphic bundle E, and for every holomorphic structure $\bar{\partial}_E$, there are trivializations of E with holomorphic transition maps such that $\bar{\partial}_E$ is the canonical holomorphic structure.*

Proof. Given a holomorphic vector bundle E, let $\bar{\partial}_E$ be defined component-wise under local frames by $\bar{\partial}_E(s) = \bar{\partial}(s^i) \otimes e_i$, where $\bar{\partial} = \pi_{0,1} \circ \mathrm{d}$. Suppose $s = s^i e_i = t^j f_j$, and $s^i = g^i_j t^j$, so $g^i_j e_i = f_j$. Then $\bar{\partial}_E(s) = \bar{\partial}(s^i) \otimes e_i = \bar{\partial}(g^i_j t^j) \otimes e_i = g^i_j \bar{\partial}(t^j) \otimes e_i = \bar{\partial}(t^j) f_j$ since $\bar{\partial} g^i_j = 0$ as E is holomorphic, thus $\bar{\partial}_E$ it is well-defined. The other direction appeals to the Newlander-Nirenberg theorem, and a proof can be found in [6]. ☐

Remark 1.5.31. For every connection ∇ on E, we have $\nabla^{1,0} : \Gamma(E) \to \Gamma(\bigwedge^{1,0} E)$ and $\nabla^{0,1} : \Gamma(E) \to \Gamma(\bigwedge^{0,1} E)$ given by $\nabla^{1,0} = \pi^{1,0} \circ \nabla$ and $\nabla^{0,1} = \pi^{0,1} \circ \nabla$ where $\pi^{p,q} : T^*_{\mathbb{C}} M \otimes E \to \bigwedge^{p,q} E$ is the projection map. The operator $\nabla^{0,1}$ is a pre-holomorphic structure. If $\nabla^{0,1}$ is a coboundary map, then E is holomorphic with canonical holomorphic structure $\bar{\partial}_E = \nabla^{0,1}$.

Proposition 1.5.32. *Given a complex vector bundle E with a holomorphic structure $\bar{\partial}_E$, and a Hermitian fibre metric h on E, there exists a unique h-compatible connection ∇ on E such that $\nabla^{0,1} = \bar{\partial}_E$.*

Proof. Fix some local frame $e_1, \ldots, e_n$, then $H = (h^i_j)$ is a matrix of functions with $h^i_j = h(e_i, e_j)$. Suppose there is a h-compatible connection ∇ such that $\nabla^{0,1} = \bar{\partial}_E$. With respect to this local frame, we have $\nabla = \mathrm{d} + A$ for some matrix of 1-forms $A = (a^i_j)$, where d act component-wise. By h-compatibility, $\mathrm{d}h(e_i, e_j) = h((\mathrm{d}+A)e_i, e_j) + h(e_i, (\mathrm{d}+A)e_j) = h(Ae_i, e_j) + h(e_i, Ae_j) = h(a^l_i e_l, e_j) + h(e_i, a^m_j e_m)$, so $\mathrm{d}h_{ij} = a^l_i h^l_j + \bar{a}^m_j h^i_m$, thus

$$\mathrm{d}H = A^\intercal H + H\bar{A}\,.$$

Decompose $\mathrm{d} = \partial + \bar{\partial}$, where $\partial, \bar{\partial} = \bar{\partial}_E$ acts component-wise, then $\nabla^{1,0} = \partial + A$ by $\nabla^{0,1} = \bar{\partial}_E$. Since ∂ maps to $(1,0)$ forms, A is a matrix of $(1,0)$-forms. Then $\partial H = A^\intercal H$ and $\bar{\partial} H = H\bar{A}$. Since $\overline{\partial H} + \overline{\bar{\partial} H} = \overline{\mathrm{d}H} = \mathrm{d}\bar{H} = \partial\bar{H} + \bar{\partial}\bar{H}$, by comparing types we get $\overline{\bar{\partial} H} = \partial \bar{H}$, so $\bar{H}A = \partial\bar{H}$. Hence

$$A = \bar{H}^{-1}(\partial\bar{H})\,.$$

This shows uniqueness. Now for existence, we define ∇ locally with the above A with respect to some local frame. Since $\partial\bar{H}$ is type $(1,0)$, we see that A is a matrix of $(1,0)$-forms. But ∂ also maps to $(1,0)$-forms, so $\nabla^{0,1} = \bar{\partial}$. Also $\bar{\partial} H = H\bar{A}$ by $\bar{H}A = \partial\bar{H}$ and $\overline{\bar{\partial} H} = \partial\bar{H}$. Since h is Hermitian, we have $H^\intercal = \bar{H}$, so $A^\intercal = (\partial\bar{H}^\intercal)(\bar{H}^\intercal)^{-1} = (\partial H)H^{-1}$, thus $\partial H = A^\intercal H$. Hence ∇ is h-compatible. By uniqueness, we obtain a well-defined connection. $\square$

Definition 1.5.33. The *Chern connection* of a pre-holomorphic structure $\bar{\partial}$ on a Hermitian vector bundle (E, h) is the unique h-compatible connection ∇ such that $\nabla^{0,1} = \bar{\partial}$.

Kähler manifolds

Next, we define and give two characterizations of Kähler manifolds.

Definition 1.5.34. Given a Riemannian metric h on an almost complex manifold (M, J), its *fundamental form* is $\omega(X, Y) = g(JX, Y)$. A *Hermitian metric on an almost complex manifold* (M, J) is Riemannian metric such that its fundamental form is J-compatible. A Hermitian metric on a complex manifold (M, J) with closed fundamental form is a *Kähler metric*, and (M, J) with a Kähler metric is a *Kähler manifold*.

Proposition 1.5.35. *A Hermitian metric on an almost complex manifold g on (M, J) is Kähler if and only if $\nabla J = 0$ for the Levi-Civita connection ∇ of h.*

Proposition 1.5.35 follows from Corollary 1.5.26.

Remark 1.5.36. A Hermitian metric on a complex manifold is Kähler if and only if there locally exists u such that $\omega = i\frac{\partial^2 u}{\partial z \partial \bar{z}}$. The functions u are called *Kähler potentials*.

Remark 1.5.37. The tangent bundle TM can be made into a complex vector bundle with scalar multiplication of i given by $iX = JX$. We can then identify TM with $T^{1,0}M$ by $X \mapsto \frac{1}{2}(X - iJX)$ as $\mathbb{C}$-vector bundles, preserving the action of J.

Proposition 1.5.38. *The holomorphic structure on the complex bundle $TM \cong T^{1,0}M$ for a complex manifold (M, J) with a Hermitian metric on a complex manifold g is given by $\bar{\partial}_X Y = \frac{1}{2}(\nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X)$, where ∇ is the Levi-Civita connection of g.*

Proof. By Proposition 1.5.30, it suffices to show that the above defined $\bar{\partial}Y$ is a TM-valued $(0,1)$-form for all Y, $\bar{\partial}$ is a linear operator satisfying Leibniz rule, and $\bar{\partial}Y = 0$ for all holomorphic Y. To show $\bar{\partial}Y$ is of type $(0,1)$, note that

$$\begin{aligned}
\bar{\partial}_{X+iJX} Y =& \bar{\partial}_X Y + i\bar{\partial}_{JX} Y \\
=& \frac{1}{2}(\nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X + i\nabla_{JX} Y + iJ\nabla_{J^2X} Y - iJ(\nabla_Y J)(JX)) \\
=& \frac{1}{2}(\nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X + i\nabla_{iX} Y + iJ\nabla_{iJX} Y - iJ(\nabla_Y J)(iX)) \\
=& \frac{1}{2}(\nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X - \nabla_X Y - J\nabla_{JX} Y + J(\nabla_Y J)(X)) \\
=& 0\,.
\end{aligned}$$

Also

$$\begin{aligned}
\bar{\partial}_X(fY) =& \frac{1}{2}(\nabla_X(fY) + J\nabla_{JX}(fY) - J(\nabla_{fY} J)X) \\
=& \frac{1}{2}((Xf)Y + f\nabla_X Y + J((JX)f)Y + fJ\nabla_{JX} Y - fJ(\nabla_Y J)X) \\
=& \frac{1}{2}((Xf)Y + i((JX)f)Y + f\nabla_X Y + fJ\nabla_{JX} Y - fJ(\nabla_Y J)X) \\
=& \frac{1}{2}((X + iJX)fY + f\nabla_X Y + fJ\nabla_{JX} Y - fJ(\nabla_Y J)X) \\
=& \bar{\partial}_X(f)Y + f\bar{\partial}_X Y\,,
\end{aligned}$$

so $\bar{\partial}$ satisfies the Leibniz rule. Since $\bar{\partial}$ is a $\mathbb{C}$-linear in the second entry and $C^\infty(U)$-linear in the first, it is an operator. Finally, if Y is holomorphic, then by Proposition 1.5.21,

$$\begin{aligned}
0 =& (\mathcal{L}_Y J)X \\
=& J[X, Y] - [JX, Y] \\
=& J\nabla_X Y - J\nabla_Y X - \nabla_{JX} Y + \nabla_Y(JX) \\
=& J\nabla_X Y - J\nabla_Y X - \nabla_{JX} Y + (\nabla_Y J)X + J\nabla_Y X \\
=& J\nabla_X Y - \nabla_{JX} Y + (\nabla_Y J)X \\
=& J\bar{\partial}_X Y\,.
\end{aligned}$$

□

Proposition 1.5.39. *Given an almost complex manifold (M, J) with a Hermitian metric on an almost complex manifold g, the Chern connection of the complex bundle TM with Hermitian metric $h = g - i\omega$ is the Levi-Civita connection of g via the identification $TM \cong T^{1,0}M$ in Remark 1.5.37, if and only if g is Kähler.*

Proof. Suppose the Levi-Civita connection ∇ is the Chern connection, then ∇ is a complex connection on TM, so $(\nabla J)Y = \nabla(JY) - J\nabla Y = \nabla(iY) - i\nabla Y = 0$. Thus $\nabla J = 0$, hence h is Kähler by Proposition 1.5.35. Suppose h is Kähler, then (M, J) is complex, and the Levi-Civita connection ∇ is g-compatible and $\nabla J = 0$. Thus

$(\nabla\omega)(X,Y) = g(J\nabla X,Y)+g(JX,\nabla Y) = g(\nabla(JX),Y)+g(JX,\nabla Y) = (\nabla g)(JX,Y)$ by $\nabla J = 0$, so $\nabla\omega = \nabla g = 0$, hence ∇ is h-compatible. It remains to show that $\nabla^{0,1} = \bar{\partial}_{TM}$. By Proposition 1.5.38 and $\nabla J = 0$,

$$\nabla^{0,1}_X Y = \nabla_{\frac{1}{2}(X+iJX)} Y = \frac{1}{2}(\nabla_X Y + i\nabla_{JX} Y) = \frac{1}{2}(\nabla_X + J\nabla_{JX}) = (\bar{\partial}_{TM})_X Y\,,$$

where X is a vector field and Y is a section of the complex bundle. □

Fubini-Study metric

The following is an important example.

Let $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ where $u \sim v$ if and only if $u = cv$ for some $c \in \mathbb{C} \setminus \{0\}$. Denote $[z_0 : \cdots : z_m] = [(z_0, \ldots, z_n)]_\sim$, and let $U_i = \{[z_0, \ldots, z_n] : z_i \neq 0\}$. Define $\phi_i : U_i \to \mathbb{C}^n$ by $\phi_i([z_0, \ldots, z_n]) = (z_0/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_n/z_i)$, then, for $i < j$,

$$(\phi_i \circ \phi_j^{-1})(w_1, \ldots, w_n) = \left(\frac{w_1}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \ldots, \frac{w_j}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \ldots \frac{w_n}{w_i}\right)\,,$$

which is holomorphic. Thus the ϕ_i defines a holomorphic structure on $\mathbb{CP}^n$, giving $\mathbb{CP}^n$ a complex structure J.

Define the projection map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ and the 2-form ρ_{FS} on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$\rho_{FS} = \frac{1}{2(\sum_m \bar{z}^m z^m)^2} \sum_{j\neq k} (\bar{z}^j z^j \mathrm{d}\bar{z}^k \wedge \mathrm{d}z^k - \bar{z}^j z^k \mathrm{d}\bar{z}^j \wedge \mathrm{d}z^k)\,,$$

and define the 2-form ω_{FS} on $\mathbb{CP}^n$, called the *Fubini-Study form*, by

$$(\omega_{FS})_x(X,Y) = (\rho_{FS})_x(u,v)$$

for $u, v \in \mathbb{C}^{n+1} = T_z(\mathbb{C}^{n+1} \setminus \{0\})$ such that $\pi(z) = x$, $\pi_* u = X$ and $\pi_* v = Y$, i.e. $\pi^*\omega_{FS} = \rho_{FS}$. The associated metric is called the *Fubini-Study metric*. One can check that ω_{FS} can be written on the open sets $\phi_j(U_j)$ as

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial} f_j\,, \qquad f_j(z) = \log\left(\frac{\sum_m \bar{z}^m z^m}{\bar{z}^j z^j}\right)\,,$$

so ω_{FS} is closed. One can also check that the restriction of ρ_{FS} to the unit sphere is the standard symplectic form $\sum_m \mathrm{d}x_m \wedge \mathrm{d}y_m$, so ω_{FS} is non-degenerate. One can moreover check that J is ω_{FS}-compatible. Hence the Fubini-Study metric is a Kähler metric.

2 Holomorphic sectional curvature

For a Kähler manifold (M,h), by [4, Prop. 4.5, Ch. IX] we have $R(X,Y)JZ = JR(X,Y)Z$, which gives

$$R(X,Y,JZ,JT) = R(X,Y,Z,T) = R(JX,JY,Z,T)\,.$$

This identity together with the symmetries and identities in Remark 1.4.4 will be used freely throughout the section.

2.1 Bisectional curvature

Holomorphic bisectional curvature is a variation of the usual sectional curvature in the setting of Kähler manifolds.

Definition 2.1.1. A plane σ in T_pM, $p \in M$ is *holomorphic* if it is J-invariant.

Definition 2.1.2. The restriction of K to holomorphic planes is called the *holomorphic sectional curvature*, and denoted H. Given a vector $X \in \sigma$ we write $H(X) = H(\sigma)$. For holomorphic planes σ, σ', the *holomorphic bisectional curvature* is defined

$$H(\sigma, \sigma') = R(X, JX, JY, Y)\,,$$

for any unit vectors $X \in \sigma$ and $Y \in \sigma'$.

Remark 2.1.3. Given a holomorphic plane σ and some non-zero (resp. unit) vector $X \in \sigma$, one can check that the pair X, JX is an orthogonal (orthonormal) basis. Recall that the sectional curvature K of a plane in the tangent space is defined to be $K(\sigma) = R(X, Y, Y, X)$ for any orthonormal basis X, Y of σ. So for a unit vector $X \in \sigma$, where σ is holomorphic, we have $H(X) = R(X, JX, JX, X)$. Since $H(\sigma, \sigma) = H(\sigma)$, and using the first Bianchi identity,

$$\begin{aligned} R(X, JX, JY, Y) =& - R(JY, X, JX, Y) - R(JX, JY, X, Y) \\ =& R(X, JY, JY, X) + R(Y, X, X, Y) = K(X, JY) + K(Y, X)\,, \end{aligned}$$

so holomorphic sectional curvature $H(-)$ has less information than holomorphic bisectional curvature $H(-,-)$, which in turn has less information than sectional curvature K.

Proposition 2.1.4 ([4, Prop. 7.3, Ch. IX])**.** *If a Kähler manifold (M, g) has constant holomorphic sectional curvature c, then*

$$\begin{aligned} R(X, Y, Z, W) =& \frac{c}{4}(-g(X, Z)g(Y, W) + g(X, W)g(Y, Z) \\ &- g(X, JZ)g(Y, JW) + g(X, JW)g(Y, JZ) - 2g(X, JY)g(Z, JW))\,, \end{aligned}$$

from which it follows that

$$R(X, JX, JY, Y) = \frac{c}{2}(g(X, X)g(Y, Y) + g(X, Y)^2 + g(X, JY)^2)\,.$$

Remark 2.1.5. Given constant holomorphic sectional curvature, let $X \in \sigma$ and $Y \in \sigma'$ be unit vectors, we have $g(Y, JY) = 0$. Then there is an orthonormal basis of T_xM of the form $X_1 = Y, X_2 = JY, X_3, \ldots, X_n$, so $1 = g(X, X) = \sum_a g(X, X_a)^2$. Thus $g(X, Y)^2 + g(X, JY)^2 \leq 1$. It follows that $R(\sigma, \sigma')$ is between $\frac{c}{2}$ and c.

Proposition 2.1.6 ([4, Prop. 7.4, Ch. IX])**.** *For any $c > 0$, the projective space $\mathbb{CP}^n$ with the Kähler form $\frac{4}{c}\omega_{FS}$ has constant holomorphic sectional curvature c, where ω_{FS} is the Fubini-Study form.*

Proposition 2.1.7 ([4, Prop. 7.9, Ch. IX])**.** *Two simply connected complete Kähler manifolds with constant holomorphic sectional curvature c are holomorphically isometric.*

Remark 2.1.8. Using Propositions 2.1.6, 2.1.7 and Synge's theorem, any connected compact Kähler manifold with positive constant sectional curvature is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini-Study metric up to a positive scalar.

We end this section with a look at complex submanifolds. Given a complex submanifold $N \subseteq M$, where (M, J, g) is a Kähler manifold, denote the induced Riemann curvature on N by R_N. Let $\mathrm{I\!I}$ be the second fundamental form of N, then from Remark 1.4.11, $\mathrm{I\!I}$ is symmetric and we have the Gauss-Codazzi formula giving

$$R_N(X, Y, Z, W) = R(X, Y, Z, W) + g(\mathrm{I\!I}(X, W), \mathrm{I\!I}(Y, Z)) - g(\mathrm{I\!I}(X, Z), \mathrm{I\!I}(Y, W)) .$$

Since $\nabla J = 0$ and $TN, T^{\perp}N$ are J-invariant, $\mathrm{I\!I}(X, JY) = (\nabla_X(JY))^{\perp} = (J\nabla_X Y)^{\perp} = J(\nabla_X Y)^{\perp} = J\mathrm{I\!I}(X, Y)$, so

$$R_N(X, JX, JY, Y) = R(X, JX, JY, Y) - \|\mathrm{I\!I}(X, Y)\|^2 - \|\mathrm{I\!I}(X, JY)\|^2 .$$

Hence the holomorphic bisectional curvature of N is less than that of M.

2.2 Frankel conjecture in dimension two

We prove Frankel's conjecture for Kähler surfaces.

Theorem 2.2.1. *Let M be a connected compact Kähler manifold with positive holomorphic bisectional curvature, and let V, W be compact complex submanifolds. If $\dim V + \dim W \geq \dim M$, then V and W have non-empty intersection.*

Proof. Suppose $V \cap W = \varnothing$. Let $\gamma : [0, l] \to M$ be a shortest geodesic between V and W, which exists since M is compact thus complete, with $\gamma(0) = p \in V$ and $\gamma(l) = q \in W$. Since γ is shortest, γ is orthogonal to T_pV and T_qW, as if γ is not orthogonal to T_pV and T_qW, we can deform it to get a shorter path. Parallel transport along γ defines a linear map from T_pV to T_qM, then by denoting its image subspace as B, B and T_qW are both orthogonal to γ since parallel transport preserves the metric. So by $\dim V + \dim W \geq \dim M$, $B \cap T_qW \neq \varnothing$. Thus we may find a unit parallel field X along γ such that $X_p \in T_pV$ and $X_q \in T_qW$. Since JX is also unit and parallel as J preserves g and $\nabla J = 0$, and $JX_p \in T_pV$, $JX_q \in T_qW$ as V, W are complex submanifolds. By Proposition 1.4.17 and denoting $T = \gamma'$, the second variation of length for γ with variational fields X and JX respectively are

$$L''_X(0) = g_q(\nabla_X X, T) - g_p(\nabla_X X, T) - \int_0^l R(T, X, X, T)\mathrm{d}t ,$$

$$L''_{JX}(0) = g_q(\nabla_{JX}(JX), T) - g_p(\nabla_{JX}(JX), T) - \int_0^l R(T, JX, JX, T)\mathrm{d}t .$$

We have

$$\begin{aligned} g(\nabla_X X + \nabla_{JX}(JX), T) =& g(J\nabla_X X + J\nabla_{JX}(JX), JT) \\ =& g(\nabla_X(JX) - \nabla_{JX} X, JT) \\ =& g([JX, X], JT) , \end{aligned}$$

where $g([JX,X],JT) = -g(J[JX,X],T) = 0$ at p and q since V,W are complex submanifolds so $[JX,X]_p \in T_pV$, $J[JX,X]_p \in T_pV$ and T is orthogonal to T_pV and similarly at q. Thus by the first Bianchi identity,

$$\begin{aligned} L''_X(0) + L''_{JX}(0) &= -\int_0^l (R(T,X,X,T) + R(T,JX,JX,T))\mathrm{d}t \\ &= -\int_0^l R(T,JT,X,JX)\mathrm{d}t < 0\,, \end{aligned}$$

since bisectional curvature is positive. Hence $L''_X(0)$ or $L''_{JX}(0)$ is negative, but γ is shortest, a contradiction. □

Starting here, we expect more sophistication in algebraic geometry, Chern classes and Hodge theory from the reader. A reference for the first is [3], and a reference for the last two is [6].

Given a local orthonormal frame $X_1,\dots,X_n$, we write $R_{abcd} = R(X_a,X_b,X_c,X_d)$, $R_{ab} = \mathrm{Ric}(X_a,X_b)$. Suppose the local frame has the form $X_1,\dots,X_n,JX_1,\dots,JX_n$, we write $R_{ii^*j^*j} = R(X_i,JX_i,JX_j,X_j)$, etc. When summing over the indices, a,b,c runs through all sections, while i,j,k only runs through the first half $X_1,\dots,X_n$.

Proposition 2.2.2. *Using the above convention, we have*

$$\mathrm{Ric}(X,Y) = \frac{1}{2}\sum_a R(X_a,JX_a,JY,X) = \sum_i R(X_i,JX_i,JY,X)\,.$$

In particular, $R_{ij} = R_{kk^*i^*j}$, so positive bisectional curvature implies positive definite Ricci curvature.

Proof. Using the first Bianchi identity,

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \sum_a R(X_a,X,Y,X_a) \\ &= \sum_a R(X_a,X,JY,JX_a) \\ &= \sum_a (-R(X,JY,X_a,JX_a) - R(JY,X_a,X,JX_a)) \\ &= \sum_a (R(JY,X,X_a,JX_a) - R(JX_a,Y,X,JX_a)) \\ &= \sum_a R(JY,X,X_a,JX_a) - \mathrm{Ric}(X,Y)\,. \end{aligned}$$

so $\mathrm{Ric}(X,Y) = \frac{1}{2}\sum_a R(X_i,JX_i,JY,X)$. □

Corollary 2.2.3. *We have* $\mathrm{Ric}(X,Y) = \mathrm{Ric}(JX,JY)$ *for any vector fields X,Y.*

Proof. Follows from Proposition 2.2.2. □

Definition 2.2.4. The Ricci form of a Kähler manifold M is defined as $\rho(-,-) = \mathrm{Ric}(J(-),-)$.

Definition 2.2.5. A $(1,1)$-form σ is *positive* if $\sigma(X, JX) > 0$ for any $X \in TM$. A holomorphic line bundle L over a complex manifold is *positive* if there exists a metric on L where the Chern connection has a curvature form Θ such that $\frac{i}{2\pi}\Theta$ is a positive $(1,1)$-form. On the other hand, L is *negative* if $L^{-1} = L^*$ is positive.

Proposition 2.2.6 ([3, p.148])**.** *Given any real closed representative σ of type $(1,1)$ for the first Chern class of a holomorphic line bundle L, there exists a metric on L such that the curvature form Θ of the Chern connection satisfies $\sigma = \frac{i}{2\pi}\Theta$. Thus L is positive if and only if its Chern connection can be represented by a positive form.*

Proposition 2.2.7 ([6])**.** *The first Chern class of $TM \cong T^{1,0}M$ for any compact Kähler manifold is represented by $\frac{1}{2\pi}\rho$, where ρ is the Ricci form.*

Proposition 2.2.8 ([6])**.** *For any complex vector bundle E over a complex manifold M of rank k, the first Chern classes of E and $\bigwedge^k E$ are the same for $k \geq 1$.*

Remark 2.2.9. For any compact Kähler manifold M of dimension $2n$ with positive bisectional curvature, let ρ be the Ricci form. Since bisectional curvature is positive, Ric is positive definite by Proposition 2.2.2, thus ρ is positive. The first Chern class of $\bigwedge^n(T^{1,0}M)$ has the representative $\frac{1}{2\pi}\rho$ by Propositions 2.2.7 and 2.2.8, so the curvature form of $\bigwedge^n(T^{1,0}M)$ is $\Theta = -i\rho$ by Propositions 2.2.6, thus $i\Theta = \rho$. Hence $\bigwedge^n(T^{1,0}M)$ is positive, so $\bigwedge^{n,0}M = (\bigwedge^n(T^{1,0}M))^*$ is negative.

Definition 2.2.10. An *algebraic Kähler manifold* is a Kähler manifold which is also projective variety. The *canonical line bundle* of an algebraic Kähler manifold M of complex dimension n is the line bundle $K_M = \bigwedge^{n,0} M$ of $(n,0)$-forms. The *anti-canonical bundle* $K_M^{-1} = K_M^*$ is the inverse line bundle of K_M, which happens to be the dual bundle. The ith *plurigenus* of M is the complex dimension $P_i = \dim \Gamma(M, K_M^i) = \dim H^0(M, K_M^i)$ of the vector space of global holomorphic sections of the ith tensor power of K_M. The *arithmetic genus* of M is $p_a = \sum_{j=0}^{n-1}(-1)^j h^{n-j,0}$, where $h^{p,q}$ denotes the Hodge numbers of M.

Definition 2.2.11. A *ruled surface* is the total space S of a holomorphic fibre bundle where the fibres are $\mathbb{CP}^1$ and the base space is a non-singular complex algebraic curve.

Lemma 2.2.12 (Castelnuovo-Andreotti, [11, Theorem 49])**.** *Given a algebraic Kähler surface M, if $P_2 = p_a = 0$, then M is either $\mathbb{CP}^2$ or a ruled surface.*

Theorem 2.2.13 (Kodaira's embedding)**.** *Given a compact Kähler manifold and a holomorphic line bundle L over M, if L is positive, then there is a holomorphic embedding of M into some complex projective space.*

Theorem 2.2.14 (Chow's theorem)**.** *A closed holomorphic submanifold of a complex projective space is an algebraic subvariety.*

Theorems 2.2.13 and 2.2.14 can be found in [6] and [7, Prop 5.1] respectively.

Theorem 2.2.15 (Kodaira-Nakano vanishing)**.** *If L is a positive line bundle over a compact Kähler manifold of complex dimension n, then $H^q(M, K_M^p \otimes L) = 0$ for $p+q > n$.*

Theorem 2.2.16 (Serre duality)**.** *If L is a holomorphic vector bundle over a compact Kähler manifold of complex dimension n, then $H^q(M, E) = H^{n-q}(M, K_M^p \otimes E^*)^*$.*

Theorems 2.2.15 and 2.2.16 can be found in [3] on p.103 and p.154 respectively. Using Serre duality and Kodaira-Nakano vanishing theorem, one obtains a dual version of Kodaira-Nakano vanishing theorem.

Theorem 2.2.17 (dual Kodaira-Nakano vanishing)**.** *If L is a negative line bundle over a compact Kähler manifold of complex dimension n, then $H^q(M, K_M^p \otimes L) = 0$ for $p+q < n$.*

We can now prove the Frankel conjecture for Kähler surfaces.

Theorem 2.2.18. *A compact Kähler surface M with positive holomorphic bisectional curvature is biholomorphically equivalent to $\mathbb{CP}^2$.*

Proof. If the bisectional curvature is positive, K_M^{-1} is positive and K_M is negative by Remark 2.2.9. Then from Kodaira's embedding theorem and Chow's theorem, M is algebraic. Using both versions of Kodaira-Nakano vanishing theorem, by letting $L = K_M^{-1}$ and $L = K_M$ respectively, we have $H^0(M, K_M^d) = 0$ for all $d \geq 0$, so the plurigenus $P_i = 0$ for $i \geq 0$ all vanishes. Since M is Kähler, it is even dimensional and orientable, and it is compact, so by Synge's theorem M is simply connected. Since it is simply connected, the first cohomology group vanishes. By Kählerity, $\dim H^1(M) = h^{1,0} + h^{0,1}$, so the Hodge number $h^{1,0} = 0$ vanishes. Note that $P_1 = \dim H^0(M, K_M) = \dim \Gamma(M, K_M)$ is the dimension of the space of global sections of $K_M = \bigwedge^{2,0} M$, thus $h^{2,0} = 0$. Then the arithmetic genus $g_a = h^{2,0} - h^{1,0} = 0$ vanishes. As $P_2 = 0$, by the surface classification theorem of Castelnuovo-Andreotti, M is either a ruled surface or $\mathbb{CP}^2$. The fibres of a ruled surface are disjoint compact complex dimension 1 submanifolds, so we eliminate this possibility with Theorem 2.2.1. Therefore M is $\mathbb{CP}^2$. □

2.3 Kähler-Einstein manifolds

We prove the Frankel conjecture when the metric is Einstein.

Lemma 2.3.1. *Given a tensor $T \in \Gamma(T^*M \otimes T^*M)$ on a Kähler manifold (M, J, g), such that $T(X,Y) = T(Y,X)$ and $T(X,Y) = T(JX, JY)$, there exists a local orthonormal frame $X_1, \dots, X_n, JX_1, \dots, JX_n$ near $x \in M$ such that $T(X_i, X_j) = 0$ at x for $i \neq j$. Moreover, we may choose any X_1 satisfying $T(X_1, -) = \lambda g(X_1, -)$ at x, for $\lambda \in \mathbb{R}$.*

Proof. Since T is symmetric, T is orthogonally diagonalizable as a linear operator on T_xM, so there exists an orthogonal basis for T_xM consisting of of T-eigenvectors. An eigenvector of T is any $X \in T_xM$ satisfying $T(X, -) = \lambda g(X, -)$ for some $\lambda \in \mathbb{R}$. So given a T-eigenvector X with eigenvalue λ, we have

$$T(JX, Y) = -T(X, JY) = -\lambda g(X, JY) = \lambda g(JX, Y)\,,$$

thus the eigenspaces of T are J-invariant. Then starting with any eigenvector X_1, we can obtain by induction a set of orthogonal basis of the form $X_1, \dots, X_n, JX_1, \dots, JX_n$ of T-eigenvector. We can then extend them to a local frame. □

Lemma 2.3.2. *For an Einstein manifold with Ricci curvature* $\mathrm{Ric}(X,Y) = kg(X,Y)$, *under an orthogonal frame, we have*

$$\sum_\alpha \frac{1}{2} \nabla_\alpha \nabla_\alpha R_{1221} = \sum_{a,b} (R_{1av2}^2 - R_{12ab}^2 + R_{1ab1} R_{2ab2}) + kR_{1221}\,.$$

Proof. Fix an orthonormal frame. By the second Bianchi identity,

$$\sum_a (\nabla_a \nabla_a R_{1221} + \nabla_a \nabla_1 R_{2a21} + \nabla_a \nabla_2 R_{a121}) = 0\,.$$

The Ricci identity given in [13] states

$$\nabla_a \nabla_r R_{sa21} = \nabla_r \nabla_a R_{sa21} - R_{arsm} R_{ma21} - R_{aram} R_{sm21} - R_{ar2m} R_{sam1} - R_{ar1m} R_{sa2m}\,.$$

The Einstein condition implies

$$R_{aram} R_{sm21} = -R_{rm} R_{sm21} = -k g_{rm} R_{sm21} = -k R_{sr21}\,,$$

and with the the second Bianchi identity

$$\nabla_a R_{sa21} = -\nabla_2 R_{sa1a} - \nabla_1 R_{saa2} = \nabla_2 R_{s1} - \nabla_1 R_{s2} = \nabla_2 k g_{s1} - \nabla_1 k g_{s2} = 0\,.$$

So plugging in $r = 1, s = 2$ and $r = 2, s = 1$ respectively,

$$\begin{aligned}
\nabla_a \nabla_1 R_{2a21} =& - R_{a12m} R_{ma21} + k R_{2121} - R_{a12m} R_{2am1} - R_{a11m} R_{2a2m}\,,\\
\nabla_a \nabla_2 R_{a121} =& R_{a21m} R_{ma21} - k R_{1221} + R_{a22m} R_{1am1} + R_{a21m} R_{1a2m}\,.
\end{aligned}$$

Combining these,

$$\begin{aligned}
\sum_a \nabla_a \nabla_a R_{1221} &= 2k R_{1221} + R_{a12m} R_{ma21} + R_{a12m} R_{2am1} + R_{a11m} R_{2a2m}\\
&\qquad\qquad - R_{a21m} R_{ma21} - R_{a22m} R_{1am1} - R_{a21m} R_{1a2m}\\
&= 2k R_{1221} + 2R_{a12m} R_{ma21} + 2R_{a12m} R_{2am1} - 2R_{a22m} R_{1am1}\\
&= 2k R_{1221} + 2(R_{am21} + R_{a21m})(R_{ma21} + R_{2am1}) - 2R_{2am2} R_{1am1}\\
&= 2k R_{1221} + 2(R_{a21m}^2 - R_{am21}^2) - 2R_{2am2} R_{1am1}\,,
\end{aligned}$$

where the second equality is by symmetries, switching the roles of indices, and the third equality by first Bianchi identity. $\square$

Lemma 2.3.3. *Let X, JX, Y, JY be orthonormal vectors and $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$, then*

$$\begin{aligned}
&H(aX + bY) + H(aX - bY) + H(aX + bJY) + H(aX - bJY)\\
=&4(a^4 H(X) + b^4 H(Y) + 4a^2 b^2 R(X, JX, JY, Y))\,.
\end{aligned}$$

Proof. By definition $H(X) = R(X, JX, JX, X)$ and $H(X, Y) = R(X, JX, JY, Y)$, so $H(JX, Y) = H(X, JY) = H(X, Y)$. Using $H(X, Y) = K(X, Y) + K(X, JY)$ in Remark 2.1.3, one computes that

$$\begin{aligned}
&H(aX + bY) + H(aX - bY)\\
=&R(aX + bY, aJX + bJY, aJX + bJY, aX + bY) +\\
&R(aX - bY, aJX - bJY, aJX - bJY, aX - bY)\\
=&2(a^4 H(X) + b^4 H(Y) + 6a^2 b^2 H(X, Y) - 4a^2 b^2 K(X, Y))\,,
\end{aligned}$$

and, replacing Y with JY, we also get

$$\begin{aligned}
&H(aX + bJY) + H(aX - bJY)\\
=&2(a^4 H(X) + b^4 H(Y) + 6a^2 b^2 H(X, Y) - 4a^2 b^2 K(X, JY))\,.
\end{aligned}$$

The result follows from $H(X, Y) = K(X, Y) + K(X, JY)$. $\square$

Lemma 2.3.4 ([12, 7.4])**.** *Given a Kähler manifold of dimension n, the scalar curvature at p is given by*

$$R(p) = \frac{n(n+1)}{\mathrm{Vol}(S^{2n-1})} \int_{S_p} H(X)\mathrm{d}X\,,$$

where S_p is the unit sphere is T_pM, $\mathrm{vol}(S^{2n-1})$ is the volume of the standard $(2n-1)$-sphere in Euclidean space, and $\mathrm{d}X$ is the canonical measure on S_p.

Theorem 2.3.5. *A n-dimensional compact connected Kähler–Einstein manifold with positive holomorphic bisectional curvature is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini-Study metric up to a positive scalar.*

Proof. Let UM denote the fibre bundle of unit tangent vectors of M. Since UM is compact, H has a maximum as a function on UM. Suppose H obtains a maximum at the unit vector $v \in T_xM$, let $H_1 = H(v)$. We see that H restricted to T_xM is the associated quadratic form of the symmetric tensor $T(-,-) = R(v, Jv, J(-), -)$. We may associate T with an operator P on T_xM given by $g(P(Y), -) = T(Y, -)$. Since T is symmetric, $g(P(X), Y) = g(X, P(Y))$. Suppose $Y \in T_xM$ is a unit vector orthogonal to v, then Y may be identified with a unit vector in $T_v(T_xM)$, so there is a curve γ on the unit sphere of T_xM passing v at 0 with $\gamma'(0) = Y$. Differentiating $H(\gamma(t)) = T(\gamma(t), \gamma(t)) = g(P(\gamma(t)), \gamma(t))$ at $t = 0$, by maximality of H at v,

$$\begin{aligned}
0 = & \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} H(\gamma(t)) \\
=& g\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} P(\gamma(t)), \gamma(0)\right) + g\left(P(\gamma(0)), \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t)\right) \\
=& g(P(Y), v) + g(Y, P(v)) \\
=& 2g(P(v), Y)\,.
\end{aligned}$$

Thus $T(v, Y) = g(P(v), Y) = 0$ for all unit Y such that $g(X, Y) = 0$, so v is an eigenvector of T, i.e. $T(v, -) = H_1 g(v, -)$. Now by applying Lemma 2.3.1 to T, we can choose an orthonormal frame $X_1, \ldots, X_n, JX_1, \ldots, JX_n$ such that $R_{11^*ai} = 0$ at x for $a \neq i^*$ and $(X_1)_x = v$. Let $k \in \mathbb{R}$ such that $\mathrm{Ric}(-,-) = kg(-,-)$. Let $Q = \left(\frac{1}{2}\sum_a \nabla_a \nabla_a R_{11^*1^*1}\right)|_x$, then by Lemma 2.3.1, with $2 = 1^*$, and evaluating everything at x,

$$\begin{aligned}
Q =& \sum_{a,b} (R^2_{1ab1^*} - R^2_{11^*ab} + R_{1ab1} R_{1^*ab1^*}) + kH_1 \\
=& \sum_{a,b\neq 1,1^*} (R^2_{1ab1^*} - R^2_{11^*ab} + R_{1ab1} R_{1^*ab1^*}) - H_1^2 + kH_1 \\
=& -2\sum_{i\neq 1} R^2_{11^*i^*i} + \sum_{a,b\neq 1,1^*} (R^2_{1ab1^*} + R_{1ab1} R_{1^*ab1^*}) - H_1^2 + kH_1 \\
=& -2\sum_{i\neq 1} R^2_{11^*i^*i} + \sum_{i,j\neq 1} ((R_{1ij1} + R_{1i^*j^*1})^2 + (R_{1ij^*1} - R_{1i^*j1})^2) - H_1^2 + kH_1 \\
\geq& -2\sum_{i\neq 1} R^2_{11^*i^*i} - H_1^2 + kH_1\,,
\end{aligned}$$

where we have used the fact that $R^2_{1ab1^*} = R^2_{1ab^*1} = R^2_{1ab1}$ after summing over b, so

$$\begin{aligned}\sum_{ab\neq 1,1^*} (R^2_{1ab1^*} + R_{1ab1}R_{1^*ab1^*}) &= \sum_{a,b\neq 1,1^*} (R^2_{1ab1} + R_{1ab1}R_{1a^*b^*1}) \\ &= \sum_{i,j\neq 1} (R^2_{1ij1} + R^2_{1i^*j^*1} + R^2_{1i^*j1} + R^2_{1ij^*1} \\ &\quad + R_{1ij1}R_{1i^*j^*1} + R_{1i^*j^*1}R_{1ij1} - R_{1i^*j1}R_{1ij^*1} - R_{1ij^*1}R_{1i^*j1}) \\ &= \sum_{i,j\neq 1} ((R_{1ij1} + R_{1i^*j^*1})^2 + (R_{1ij^*1} - R_{1i^*j1})^2) \,.\end{aligned}$$

But $k = R_{11} = \sum_i R_{11^*i^*i} = H_1 + \sum_{i\neq 1} R_{11^*i^*i}$ at x by Proposition 2.2.2, so

$$Q \geq -2\sum_{i\neq 1} R^2_{11^*i^*i} - H_1^2 + \left(H_1 + \sum_{i\neq 1} R_{11^*i^*i}\right) H_1 = \sum_{i\neq 1} R_{11^*i^*i}(H_1 - 2R_{11^*i^*i}) \,.$$

For each $i \neq 1$, let $aX + bY \in T_xM$ be a unit vector where $X = X_1$ and $Y = X_i$. Using Lemma 2.3.3 and maximality of H_1,

$$4H_1 \geq 4(a^4H(X) + b^4H(Y) + 4a^2b^2R_{11^*i^*i}) \,,$$

thus moving the first term over,

$$(1 + a^2)b^2H_1 = (1 + a^2)(1 - a^2)H_1 \geq b^4H(Y) + 4a^2b^2R_{1i^*i^*1} \,,$$

so $H_1 \geq 2R_{11^*i^*i}$ by cancelling b^2 and setting $a = 1$. Hence $Q \geq 0$. But $Q \leq 0$ by maximality of H_1, hence $Q = 0$. Since $R_{11^*i^*i} > 0$ we have $H_1 = 2R_{11^*i^*i}$. Thus $k = H_1 + \frac{1}{2}(n-1)H_1 = \frac{1}{2}(n+1)H_1$. Then for each $p \in M$, the scalar curvature is $R(p) = R_{aa} = 2nk = n(n+1)H_1$, combining this with Lemma 2.3.4,

$$\int_{S_p} (H_1 - H(X))\mathrm{d}X = 0 \,,$$

so $H(X) = H_1 > 0$ for all $X \in U_pM$. The result follows from Remark 2.1.8. □

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