1. **Swap test (40%).** Let $\text{SWAP} : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ be the swap operator, which swaps the two registers, acting as $\text{SWAP} |\psi\rangle |\phi\rangle = |\phi\rangle |\psi\rangle$ for any pair of pure states $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$.

The swap test is a way of testing whether two states are the same by measuring the eigenspaces of $\text{SWAP}$. It declares them to be the same if the outcome is the $+1$ eigenspace, and different if the outcome is the $-1$ eigenspace.

(a) (10%) Show that the eigenvalues of $\text{SWAP}$ are $\pm 1$, i.e. $\text{SWAP} = P_+ - P_-$ for projectors $P_+$ and $P_-$.

**Solution.** $\text{SWAP}$ is a unitary matrix satisfying $\text{SWAP}^2 = I_{d^2}$, so its eigenvalues are complex numbers $\lambda$ satisfying $\lambda^2 = 1$. Therefore, its eigenvalues are in the set $\{1, -1\}$. Both of these are eigenvalues, since otherwise $\text{SWAP}$ equals $I_{d^2}$ or $-I_{d^2}$.

(b) (10%) What is the probability that the test declares two pure states $|\psi\rangle$ and $|\phi\rangle$ to be the same?

**Solution.** The projector $P_+ = \frac{1}{2}(I + \text{SWAP})$ onto the $+1$ eigenspace of $\text{SWAP}$ is the POVM element giving the probability that the test declares the states to be the same. The probability it declares two density matrices $\rho$ and $\sigma$ to be the same is $\text{Tr} P_+ (\rho \otimes \sigma)$, where $P_+ = \frac{1}{2} \text{Tr}(I + \text{SWAP})$. We can evaluate this trace exactly, using the identities $\text{Tr} \rho \otimes \sigma = 1$ and

$$\text{Tr}(\rho \otimes \sigma)_{\text{SWAP}} = \sum_{ij} ((\rho \otimes \sigma)_{\text{SWAP}})_{ij} = \sum_{ij} \rho_{ij} \sigma_{ji} = \text{Tr}(\rho \sigma),$$

giving $\text{Tr} P_+ (\rho \otimes \sigma) = \frac{1+\text{Tr} \rho \sigma}{2}$. Therefore, the probability it declares two pure states $|\psi\rangle$ and $|\phi\rangle$ to be the same is $\frac{1+|\langle\psi|\phi\rangle|^2}{2}$.

(c) (20%) Consider the case for qubits ($d = 2$). What is the probability that the SWAP test declares two density matrices

$$\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z), \quad \sigma = \frac{1}{2}(I + r'_x X + r'_y Y + r'_z Z)$$

to be the same?

**Solution.** Using the probability $\frac{1+\text{Tr} \rho \sigma}{2}$ derived in the last section, we need only compute $\text{Tr} \rho \sigma$:

$$\text{Tr} \rho \sigma = \frac{1}{2}(1 + r_x r'_x + r_y r'_y + r_z r'_z) = \frac{1}{2}(1 + r \cdot r')$$

(other terms such as $r_x r'_y$ vanish because e.g. $\text{Tr} X Y = 0$). Therefore the probability it declares the qubit states to be the same is

$$\text{Tr} P_+ (\rho \otimes \sigma) = \frac{1+1+rr'}{2} = \frac{3 + r \cdot r'}{4}.$$
2. Entanglement of the Werner state (30%)

Let
\[
\rho := (1 - p) \left( |\phi^+ \rangle \langle \phi^+ | + |\phi^- \rangle \langle \phi^- | + |\psi^+ \rangle \langle \psi^+ | \right) \frac{1}{3} + p |\psi^- \rangle \langle \psi^- |
\]
be the Werner state discussed in class.

(a) (10%) Write \(\rho\) as a 4 \times 4 matrix.

Solution.
\[
\rho = \frac{1 - p}{3} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} + p \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{1 - p}{3} & 0 & 0 & 0 \\
0 & \frac{1 + 2p}{6} & \frac{1 - 4p}{6} & 0 \\
0 & \frac{1 - 4p}{6} & \frac{1 + 2p}{6} & 0 \\
1 - 4p & 0 & 0 & \frac{1 - p}{3}
\end{pmatrix}
\]

(b) (20%) Using the partial transpose test, show that \(\rho\) is separable if \(p \leq 1/2\) and is entangled if \(p > 1/2\).

Solution. In class, we discussed (but did not prove) the fact that the partial transpose test is necessary and sufficient to detect entanglement between two qubits. The partial transpose of \(\rho\) is
\[
\begin{pmatrix}
\frac{1 - p}{3} & 0 & 0 & \frac{1 - 4p}{6} \\
0 & \frac{1 + 2p}{6} & \frac{1 - 4p}{6} & 0 \\
0 & \frac{1 - 4p}{6} & \frac{1 + 2p}{6} & 0 \\
\frac{1 - 4p}{6} & 0 & 0 & \frac{1 - p}{3}
\end{pmatrix}.
\]

This matrix has two distinct eigenvalues: \(\frac{1 + 2p}{6}\) and \(\frac{1 - 2p}{2}\), the latter of which is positive iff \(p \leq \frac{1}{2}\).
3. Symmetric tomography (30%)

Suppose that $E_1, \ldots, E_{d^2}$ is an informationally complete POVM on a $\mathbb{C}^d$. This means that the map $\rho \mapsto \{ p(j) = \text{Tr} \rho E_j : j = 1, \ldots, d^2 \}$ has an inverse, allowing a reconstruction of the form

$$\rho = \sum_{j=1}^{d^2} a_j E_j,$$

where the $a_j$ are related to the measurement probabilities $p(j) = \text{Tr} E_j \rho$ by an affine map. Repeated measurements give an estimate of the $p(j)$, which in turn give an estimate of the density matrix $\rho$. In general, finding the $a_j$ from the $p(j)$ involves inverting a (possibly complicated) $d^2 \times d^2$ matrix.

Further assume that the POVM is also symmetric, in the sense that $\text{Tr} E_i = \frac{1}{d}$, $\text{Tr} E_i^2 = b$ and $\text{Tr} E_i E_j = c$ for some $b, c$. In this case, show that for each $j$, the coefficient $a_j$ only depends on $p(j)$ via a simple formula.

**Solution.** Assume that $\rho = \sum_j a_j E_j$. We will show how to reconstruct each $a_j$ from $p(j)$. First note that

$$p(j) = \text{Tr} \rho E_j = \text{Tr} \left( \sum_i a_i E_i \right) E_j = \sum_i a_i \text{Tr} E_i E_j = a_j \text{Tr} E_j^2 + \sum_{i \neq j} a_i \text{Tr} E_i E_j = a_j b + c \sum_{i \neq j} a_i.$$

On the other hand,

$$1 = \text{Tr} \rho = \text{Tr} \sum_i a_i E_i = \sum_i a_i \text{Tr} E_i = \sum_i a_i / d,$$

so the coefficients $a_j$ satisfy $\sum a_j = d$. Therefore, $\sum_{i \neq j} a_i = d - a_j$. Substituting this into the above gives

$$p(j) = a_j b + c(d - a_j) = a_j (b - c) + cd,$$

so the coefficients $a_j$ can be recovered from the probabilities $p(j)$ as

$$a_j = \frac{p(j) - cd}{b - c}.$$