QIC 710 / CO 681 / AMATH 871 / CS 768 / PHYS 767 Fall 2018

Homework 5 Solutions

1. **Swap test (40%).** Let $\mathrm{SWAP}\colon \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ be the swap operator, which swaps the two registers, acting as $\mathrm{SWAP}|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$ for any pair of pure states $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$.

   The swap test is a way of testing whether two states are the same by measuring the eigenspaces of SWAP. It declares them to be the same if the outcome is the $+1$ eigenspace, and different if the outcome is the -1 eigenspace.

   (a) (10%) Show that the eigenvalues of SWAP are ± 1, i.e. $\mathrm{SWAP} = P_+ - P_-$ for projectors P_+ and P_-.

   Solution. SWAP is a unitary matrix satisfying $\mathrm{SWAP}^2 = I_{d^2}$, so its eigenvalues are complex numbers λ satisfying $\lambda^2 = 1$. Therefore, its eigenvalues are in the set $\{1, -1\}$. Both of these are eigenvalues, since otherwise SWAP equals I_{d^2} or $-I_{d^2}$.

   (b) (10%) What is the probability that the test declares two pure states $|\psi\rangle$ and $|\phi\rangle$ to be the same?

   Solution. The projector $P_+ = \frac{1}{2}(I + \mathrm{SWAP})$ onto the $+1$ eigenspace of SWAP is the POVM element giving the probability that the test declares the states to be the same. The probability it declares two density matrices ρ and σ to be the same is $\operatorname{Tr} P_+(\rho \otimes \sigma)$, where $P_+ = \frac{1}{2}\operatorname{Tr}(I + \mathrm{SWAP})$. We can evaluate this trace exactly, using the identities $\operatorname{Tr} \rho \otimes \sigma = 1$ and

   $$\operatorname{Tr}(\rho \otimes \sigma)\mathrm{SWAP} = \sum_{ij} ((\rho \otimes \sigma)\mathrm{SWAP})_{ij} = \sum_{ij} \rho_{ij}\sigma_{ji} = \operatorname{Tr}(\rho\sigma),$$

   giving $\operatorname{Tr} P_+(\rho \otimes \sigma) = \frac{1+\operatorname{Tr}\rho\sigma}{2}$. Therefore, the probability it declares two pure states $|\psi\rangle$ and $|\phi\rangle$ to be the same is $\frac{1+|\langle\psi|\phi\rangle|^2}{2}$.

   (c) (20%) Consider the case for qubits ($d = 2$). What is the probability that the SWAP test declares two density matrices

   $$\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z), \quad \sigma = \frac{1}{2}(I + r'_x X + r'_y Y + r'_z Z)$$

   to be the same?

   Solution. Using the probability $\frac{1+\operatorname{Tr}\rho\sigma}{2}$ derived in the last section, we need only compute $\operatorname{Tr}\rho\sigma$:

   $$\operatorname{Tr}\rho\sigma = \frac{1}{2}(1 + r_x r'_x + r_y r'_y + r_z r'_z) = \frac{1}{2}(1 + r \cdot r')$$

   (other terms such as $r_x r'_y$ vanish because e.g. $\operatorname{Tr} XY = 0$). Therefore the probability it declares the qubit states to be the same is

   $$\operatorname{Tr} P_+(\rho \otimes \sigma) = \frac{1 + \frac{1+r\cdot r'}{2}}{2} = \frac{3 + r \cdot r'}{4}.$$

2. **Entanglement of the Werner state (30%)**

Let

$$\rho := (1-p)\frac{|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|}{3} + p|\psi^-\rangle\langle\psi^-|$$

be the Werner state discussed in class.

(a) (10%) Write ρ as a 4×4 matrix.

Solution.

$$\rho = \frac{1-p}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + p\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1-p}{3} & 0 & 0 & 0 \\ 0 & \frac{1+2p}{6} & \frac{1-4p}{6} & 0 \\ 0 & \frac{1-4p}{6} & \frac{1+2p}{6} & \\ 0 & 0 & 0 & \frac{1-p}{3} \end{pmatrix}$$

(b) (20%) Using the partial transpose test, show that ρ is separable if $p \leq 1/2$ and is entangled if $p > 1/2$.

Solution. In class, we discussed (but did not prove) the fact that the partial transpose test is necessary and sufficient to detect entanglement between two qubits. The partial transpose of ρ is

$$\begin{pmatrix} \frac{1-p}{3} & 0 & 0 & \frac{1-4p}{6} \\ 0 & \frac{1+2p}{6} & 0 & 0 \\ 0 & 0 & \frac{1+2p}{6} & \\ \frac{1-4p}{6} & 0 & 0 & \frac{1-p}{3} \end{pmatrix}.$$

This matrix has two distinct eigenvalues: $\frac{1+2p}{6}$ and $\frac{1-2p}{2}$, the latter of which is positive iff $p \leq \frac{1}{2}$.

3. **Symmetric tomography (30%)**

Suppose that $E_1, \ldots, E_{d^2}$ is an informationally complete POVM on a $\mathbb{C}^d$. This means that the map $\rho \mapsto \{p(j) = \operatorname{Tr} \rho E_j : j = 1, \ldots, d^2\}$ has an inverse, allowing a reconstruction of the form

$$\rho = \sum_{j=1}^{d^2} a_j E_j,$$

where the a_j are related to the measurement probabilities $p(j) = \operatorname{Tr} E_j \rho$ by an affine map. Repeated measurements give an estimate of the $p(j)$, which in turn give an estimate of the density matrix ρ. In general, finding the a_j from the $p(j)$ involves inverting a (possibly complicated) $d^2 \times d^2$ matrix.

Further assume that the POVM is also symmetric, in the sense that $\operatorname{Tr} E_i = \frac{1}{d}$, $\operatorname{Tr} E_i^2 = b$ and $\operatorname{Tr} E_i E_j = c$ for some b, c. In this case, show that for each j, the coefficient a_j only depends on $p(j)$ via a simple formula.

Solution. Assume that $\rho = \sum_j a_j E_j$. We will show how to reconstruct each a_j from $p(j)$. First note that

$$p(j) = \operatorname{Tr} \rho E_j = \operatorname{Tr} \left(\sum_i a_i E_i \right) E_j = \sum_i a_i \operatorname{Tr} E_i E_j = a_j \operatorname{Tr} E_j^2 + \sum_{i \neq j} a_i \operatorname{Tr} E_i E_j = a_j b + c \sum_{i \neq j} a_i.$$

On the other hand,

$$1 = \operatorname{Tr} \rho = \operatorname{Tr} \sum_i a_i E_i = \sum_i a_i \operatorname{Tr} E_i = \sum_i a_i / d,$$

so the coefficients a_j satisfy $\sum a_j = d$. Therefore, $\sum_{i \neq j} a_i = d - a_j$. Substituting this into the above gives

$$p(j) = a_j b + c(d - a_j) = a_j (b - c) + cd,$$

so the coefficients a_j can be recovered from the probabilities $p(j)$ as

$$a_j = \frac{p(j) - cd}{b - c}.$$