QIC 710 / CO 681 / AMATH 871 / CS 768 / PHYS 767 Fall 2018

Homework 1 Solutions

1. **Single-qubit quantum circuits (20%)** Suppose you have access to a single-qubit quantum computer that can perform, with essentially perfect accuracy, any of the following three gates:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_8 \end{pmatrix},$$

where $\zeta_8 = e^{2\pi i/8}$. Then the computer can also perfectly perform any sequence of these gates, for instance

$$-\boxed{U_1}-\boxed{U_2}-\boxed{U_3}-\boxed{U_4}- \quad = U_4U_3U_2U_1,$$

where $U_i \in \{X, H, T\}$. Note that the order is *reversed* between the circuit notation and matrix multiplication, i.e. matrices act on vectors from the left, whereas time moves to the right in the circuit.

Write each matrix below as a product of some number of gates from the set $\{X, H, T\}$. You do not have to draw the circuit.

Solutions:

(a) $Z = HXH$

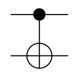
(b) $\begin{pmatrix} \zeta_8 & 0 \\ 0 & 1 \end{pmatrix} = XTX$

(c) $\frac{1}{\sqrt{2}}\begin{pmatrix} -\zeta_8 & \zeta_8 \\ 1 & 1 \end{pmatrix} = XTHX$

(d) $\frac{1}{2}\begin{pmatrix} 1+\zeta_8 & \zeta_8 - i \\ 1-\zeta_8 & \zeta_8 + i \end{pmatrix} = HTHT$

2. **CNOT and the Bell basis (20%)**

Recall from class that the controlled-NOT gate, or CNOT, implements the following unitary in the computational basis:

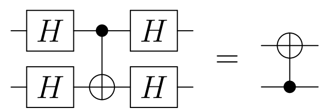


$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Give a set of eigenvectors and corresponding eigenvalues of CNOT.

Solution. Eigenvalue -1 with eigenvector $|1\rangle|-\rangle$ and eigenvalue $+1$ with 3-dimensional eigenspace spanned by $|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|+\rangle$.

(b) Show that



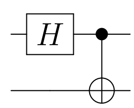
and write down the corresponding 4×4 unitary matrix. This shows that in the $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$ basis, a CNOT acts as if the second qubit controls the first.

Solution. We can calculate with block matrices:

$$\begin{pmatrix} H & H \\ H & -H \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} H & H \\ H & -H \end{pmatrix} = \begin{pmatrix} I+Z & I-Z \\ I-Z & I+Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

This unitary preserves $|00\rangle$ and $|10\rangle$, while swapping $|01\rangle$ and $|11\rangle$, i.e. it flips the first bit iff the second bit as required.

(c) Write down the 4×4 unitary matrix implemented by the following circuit:



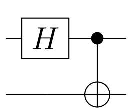
You should find that this unitary transforms the computational basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ to the Bell basis:

$$|\Psi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, |\Psi_-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Phi_+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, |\Phi_-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Which Bell state does each computational basis state get mapped to?

Solution.



$$= \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ X & -X \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

$$|00\rangle \mapsto |\Psi_+\rangle, \quad |01\rangle \mapsto |\Phi_+\rangle, \quad |10\rangle \mapsto |\Psi_-\rangle, \quad |11\rangle \mapsto |\Phi_-\rangle.$$

3. **Measuring individual qubits (20%)** Consider the following pure state of two qubits:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|11\rangle.$$

(a) Suppose the first qubit is measured in the computational basis. Compute the probability of each outcome and the corresponding "collapsed" post-measurement states by finding probabilities $p_0 + p_1 = 1$ and normalized states $|\psi_0\rangle$ and $|\psi_1\rangle$ such that

$$|\psi\rangle = \sqrt{p_0}|0\rangle|\psi_0\rangle + \sqrt{p_1}|1\rangle|\psi_1\rangle.$$

Solution.

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{3}}|0\rangle(|0\rangle + |1\rangle) + \frac{1}{\sqrt{3}}|1\rangle|1\rangle \\ &= \sqrt{\frac{2}{3}}|0\rangle|+\rangle + \frac{1}{\sqrt{3}}|1\rangle|1\rangle. \end{aligned}$$

Therefore, measuring the first qubit in the computational basis results in 0 or 1 with probabilities $p_0 = 2/3$, $p_1 = 1/3$, with corresponding post-measurement states $|\psi_0\rangle = |+\rangle$ and $|\psi_1\rangle = |1\rangle$.

(b) Compute the probabilities and post-measurement states associated with a measurement in the $|\pm\rangle$ basis by finding probabilities $p_+ + p_- = 1$ and normalized states $|\psi_+\rangle$ and $|\psi_-\rangle$ such that

$$|\psi\rangle = \sqrt{p_+}|+\rangle|\psi_+\rangle + \sqrt{p_-}|-\rangle|\psi_-\rangle.$$

Solution. Rewriting the first qubit in the $|\pm\rangle$ basis gives

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{3}}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)|0\rangle + \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)|1\rangle + \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)|1\rangle \\ &= \sqrt{\frac{5}{6}}|+\rangle\left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right) + \frac{1}{\sqrt{6}}|-\rangle|0\rangle \end{aligned}$$

Therefore, measuring the first qubit in the computational basis results in 0 or 1 with probabilities $p_+ = 5/6$, $p_- = 1/6$, with corresponding post-measurement states $|\psi_+\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$ and $|\psi_-\rangle = |0\rangle$.

4. **Distinguishing nondistinguishable states (20%)**

Recall from class that two states $|\psi_0\rangle$ and $|\psi_1\rangle$ are perfectly distinguishable iff they are orthogonal. When they are orthogonal, they are distinguished by a measurement in the basis $|\psi_0\rangle$, $|\psi_1\rangle$. Otherwise, we may try to distinguish them as well as possible by performing a measurement in some orthonormal basis $|\phi_0\rangle, |\phi_1\rangle$. For each i, this measurement successfully identifies the state $|\psi_i\rangle$ with probability $|\langle\phi_i|\psi_i\rangle|^2$.

Suppose further that with probability p_0, $|\psi_0\rangle$ is prepared and, with probability p_1, $|\psi_1\rangle$ is prepared. Then measuring in the $|\phi_0\rangle, |\phi_1\rangle$ basis successfully identifies i with probability

$$\Pr(\text{correct}) = |\langle\phi_0|\psi_0\rangle|^2 p_0 + |\langle\phi_1|\psi_1\rangle|^2 p_1.$$

Maximizing Pr(correct) over orthonormal measurement bases $|\phi_0\rangle, |\phi_1\rangle$ gives a measure of the distinguishability of the probabilistic preparation $(p_0, |\psi_0\rangle), (p_1, |\psi_1\rangle)$.

(a) Suppose that

$$|\psi_0\rangle = |0\rangle, |\psi_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

are prepared with equal probabilities $p_0 = p_1 = \frac{1}{2}$. Find the maximum value of Pr(correct) over all measurement bases $|\phi_0\rangle, |\phi_1\rangle$ and give the optimal basis.

Note that because the $|\langle\phi_i|\psi_i\rangle|^2$ are invariant under multiplying $|\phi_i\rangle$ by a phase, it is sufficient to optimize over bases of the form

$$|\phi_0\rangle = \alpha x|0\rangle - \sqrt{1-x^2}|1\rangle, \ \ |\phi_1\rangle = \sqrt{1-x^2}|0\rangle + \bar{\alpha}x|1\rangle,$$

where $0 \le x \le 1$ and $|\alpha|^2 = 1$.

Solution. First we calculate

$$\begin{aligned}
|\langle\phi_0|\psi_0\rangle|^2 &= x^2 \\
|\langle\phi_1|\psi_1\rangle|^2 &= \left|\frac{\sqrt{3}}{2}\sqrt{1-x^2} + \frac{\bar{\alpha}x}{2}\right|^2 = \frac{3}{4}(1-x^2) + \frac{3}{4}x\sqrt{1-x^2}(\alpha+\bar{\alpha}) + \frac{1}{4}x^2 \\
&\le \frac{3}{4}(1-x^2) - \frac{1}{2}x^2 + \frac{\sqrt{3}}{2}x\sqrt{1-x^2} \\
&= \frac{3}{4} - \frac{1}{2}x^2 + \frac{\sqrt{3}}{2}x\sqrt{1-x^2}
\end{aligned}$$

with equality achieved for $\alpha = 1$. Therefore,

$$\Pr(\text{correct}) = \frac{3}{8} + \frac{1}{4}x^2 + \frac{\sqrt{3}}{4}x\sqrt{1-x^2}.$$

By the derivative test or with software like Mathematica, one can show that this probability is maximized at $x = \frac{\sqrt{3}}{2}$, so the optimal probability and bases are given by

$$\Pr(\text{correct}) = \frac{3}{4}, \ \ |\phi_0\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle, \ \ |\phi_1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

(b) Same question, only with $p_0 = \frac{3}{4}, p_1 = \frac{1}{4}$. Therefore,

$$\Pr(\text{correct}) = \frac{3}{16} + \frac{5}{8}x^2 + \frac{\sqrt{3}}{8}x\sqrt{1-x^2}.$$

This probability is maximized for $x = \sqrt{\frac{1}{2} + \frac{5}{4\sqrt{7}}} \approx .9861$, with optimal probability and measurement basis given by

$$\Pr(\text{correct}) = \frac{1}{2} + \frac{\sqrt{7}}{8} \approx .8307,$$

$$\begin{aligned}
|\phi_0\rangle &= \sqrt{\frac{1}{2} + \frac{5}{4\sqrt{7}}}|0\rangle - \sqrt{\frac{1}{2} - \frac{5}{4\sqrt{7}}}|1\rangle \approx .9861|0\rangle - .1660|1\rangle \\
|\phi_1\rangle &= \sqrt{\frac{1}{2} - \frac{5}{4\sqrt{7}}}|0\rangle + \sqrt{\frac{1}{2} + \frac{5}{4\sqrt{7}}}|1\rangle \approx .1660|0\rangle + .9861|1\rangle.
\end{aligned}$$

Note that when one of the states is more likely, the average success probability increases.

5. **Entanglement swapping (20%).** Consider four qubits, labeled 1,2,3,4, where qubits 1 and 2 are in the Bell state $|\Psi^+\rangle$, and qubits 3 and 4 also in the Bell state $|\Psi^+\rangle$. The state of all four qubits is then given by the tensor product $|\Psi^+\rangle|\Psi^+\rangle$. Here we will see what happens if we perform a Bell measurement on qubits 2 and 3.

   (a) Let U be the unitary that moves the 4th qubit into the second position, defined by $U|abcd\rangle = |adbc\rangle$. Show that

$$U|\Psi^+\rangle|\Psi^+\rangle = \frac{1}{2}|\Psi^+\rangle|\Psi^+\rangle + \frac{1}{2}|\Psi^-\rangle|\Psi^-\rangle + \frac{1}{2}|\Phi^+\rangle|\Phi^+\rangle + \frac{1}{2}|\Phi^-\rangle|\Phi^-\rangle.$$

   Solution. For each of the four Bell states, we can compute their tensor products:

$$\begin{aligned}
|\Psi^+\rangle|\Psi^+\rangle &= \frac{1}{2}(|00\rangle + |11\rangle)^{\otimes 2} = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle) \\
|\Psi^-\rangle|\Psi^-\rangle &= \frac{1}{2}(|00\rangle - |11\rangle)^{\otimes 2} = \frac{1}{2}(|0000\rangle - |0011\rangle - |1100\rangle + |1111\rangle) \\
|\Phi^+\rangle|\Phi^+\rangle &= \frac{1}{2}(|01\rangle + |10\rangle)^{\otimes 2} = \frac{1}{2}(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle) \\
|\Phi^-\rangle|\Phi^-\rangle &= \frac{1}{2}(|01\rangle - |10\rangle)^{\otimes 2} = \frac{1}{2}(|0101\rangle - |0110\rangle - |1001\rangle + |1010\rangle).
\end{aligned}$$

   Therefore, the right-hand-side of the equation to be shown is

$$\frac{1}{2}(|0000\rangle + |1111\rangle + |0101\rangle + |1010\rangle)$$

   On the other hand, the left-hand-side is equal to

$$\begin{aligned}
U|\Psi^+\rangle|\Psi^+\rangle &= \frac{1}{2}(U|0000\rangle + U|0011\rangle + U|1100\rangle + U|1111\rangle) \\
&= \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)
\end{aligned}$$

   as required.

   (b) If one does a Bell measurement on qubits 2 and 3 of the state $|\Psi^+\rangle|\Psi^+\rangle$, what are the probabilities and corresponding post-measurement states of qubits 1 and 4 for each possible outcome of the Bell measurement?

   Solution. Because U only permutes the qubits, this is the same as asking for the post-measurement states of the first two qubits of $U|\Psi^+\rangle|\Psi^+\rangle$ upon measuring the last two qubits. The result of Part (a) thus tells us that the remaining qubits will be in the same Bell state that the measured ones are found to be in.