QIC 710 / CO 681 / AMATH 871 / CS 768 / PHYS 767 Fall 2017

Homework 5

Optional assignment - Due date: Monday, Dec 10 (or later with permission).

This assignment is optional. To compute your overall homework grade, I will average the top 4 out of 5 homework assignments. If your final grade is borderline, having completed all 5 homeworks may help raise your final grade.

Completed homeworks will be submitted online in Crowdmark. You will receive an email with instructions closer to the due date. A requirement for this is that each of your solutions *must begin on a new page.* Typesetting solutions in LaTeX is recommended but not required. If you do write your solutions by hand, please ensure that your scans are legible before you upload them.

1. **Swap test (40%).** Let SWAP: $\mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ be the swap operator, which swaps the two registers, acting as $\mathrm{SWAP}|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$ for any pair of pure states $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$.

   The swap test is a way of testing whether two states are the same by measuring the eigenspaces of SWAP. It declares them to be the same if the outcome is the $+1$ eigenspace, and different if the outcome is the -1 eigenspace.

   (a) (10%) Show that the eigenvalues of SWAP are ± 1, i.e. $\mathrm{SWAP} = P_+ - P_-$ for projectors P_+ and P_-.

   (b) (10%) What is the probability that the test declares two pure states $|\psi\rangle$ and $|\phi\rangle$ to be the same?

   (c) (20%) Consider the case for qubits ($d = 2$). What is the probability that the SWAP test declares two density matrices

   $$\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z), \;\; \sigma = \frac{1}{2}(I + r'_x X + r'_y Y + r'_z Z)$$

   to be the same?

2. **Entanglement of the Werner state (30%)**

   Let
   $$\rho := (1-p)\frac{|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|}{3} + p|\psi^-\rangle\langle\psi^-|$$
   be the Werner state discussed in class.

   (a) (10%) Write ρ as a 4×4 matrix.

   (b) (20%) Using the partial transpose test, show that ρ is separable if $p \leq 1/2$ and is entangled if $p > 1/2$.

3. **Symmetric tomography (30%)**

Suppose that $E_1, \dots, E_{d^2}$ is an informationally complete POVM on a $\mathbb{C}^d$. This means that the map $\rho \mapsto \{p(j) = \operatorname{Tr} \rho E_j : j = 1, \dots, d^2\}$ has an inverse, allowing a reconstruction of the form

$$\rho = \sum_{j=1}^{d^2} a_j E_j,$$

where the a_j are related to the measurement probabilities $p(j) = \operatorname{Tr} E_j \rho$ by an affine map. Repeated measurements give an estimate of the $p(j)$, which in turn give an estimate of the density matrix ρ. In general, finding the a_j from the $p(j)$ involves inverting a (possibly complicated) $d^2 \times d^2$ matrix.

Further assume that the POVM is also symmetric, in the sense that $\operatorname{Tr} E_i = \frac{1}{d}$, $\operatorname{Tr} E_i^2 = b$ and $\operatorname{Tr} E_i E_j = c$ for some b, c. In this case, show that for each j, the coefficient a_j only depends on $p(j)$ via a simple formula.