Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lecture 9 (2017)

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More state distinguishing problems

More state distinguishing problems

Which of these states are distinguishable? Divide them into equivalence classes:

1. $|0\rangle + |1\rangle$

2. $-|0\rangle - |1\rangle$

3. $\begin{cases} |0\rangle & \text{with prob. } ½ \\ |1\rangle & \text{with prob. } ½ \end{cases}$

4. $\begin{cases} |0\rangle + |1\rangle & \text{with prob. } ½ \\ |0\rangle - |1\rangle & \text{with prob. } ½ \end{cases}$

5. $\begin{cases} |0\rangle & \text{with prob. } ½ \\ |0\rangle + |1\rangle & \text{with prob. } ½ \end{cases}$

6. $\begin{cases} |0\rangle & \text{with prob. } ¼ \\ |1\rangle & \text{with prob. } ¼ \\ |0\rangle + |1\rangle & \text{with prob. } ¼ \\ |0\rangle - |1\rangle & \text{with prob. } ¼ \end{cases}$

7. The first qubit of $|01\rangle - |10\rangle$

This is a probabilistic mixed state

Answers later on ...

Density matrix formalism

Density matrices (1)

Until now, we've represented quantum states as ***vectors*** (e.g. $|\psi\rangle$, and all such states are called ***pure states***).

An alternative way of representing quantum states is in terms of ***density matrices*** (a.k.a. ***density operators***).

The density matrix of a pure state $|\psi\rangle$ is the matrix $\rho = |\psi\rangle\langle\psi|$

Example: the density matrix of $\alpha|0\rangle + \beta|1\rangle$ is

$$\rho = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\alpha^* \quad \beta^*) = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

Density matrices (2)

How do quantum operations work using density matrices?

Effect of a unitary operation on a density matrix

Applying U to ρ yields $U\rho U^\dagger$.

This is because the modified state is $U|\psi\rangle\langle\psi|U^\dagger$.

Effect of a measurement on a density matrix

Measuring state ρ with respect to the basis $|\varphi_1\rangle$, $|\varphi_2\rangle$,…, $|\varphi_d\rangle$ yields the kth outcome with probability $\langle\varphi_k|\rho|\varphi_k\rangle$.

This is because $\langle\varphi_k|\rho|\varphi_k\rangle = \langle\varphi_k|\psi\rangle\langle\psi|\varphi_k\rangle = |\langle\varphi_k|\psi\rangle|^2$.

After the measurement, the state collapses to $|\varphi_k\rangle\langle\varphi_k|$.

Density matrices (3)

A probability distribution on pure states is called a ***mixed state***:

$((|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), \ldots (|\psi_m\rangle, p_m))$

The ***density matrix*** associated with such a mixed state is

$$\rho = \sum_{k=1}^{m} p_k |\psi_k\rangle\langle\psi_k|$$

Example: the density matrix for $\big((|0\rangle, 1/2), (|1\rangle, 1/2)\big)$ is

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Question: what is the density matrix of

$\big((|0\rangle + |1\rangle, 1/2), (|0\rangle - |1\rangle, 1/2)\big)$?

Density matrices (4)

How do quantum operations work for these ***mixed*** states?

Effect of a unitary operation on a density matrix:

applying U to ρ ***still*** gives $U\rho U^\dagger$.

This is because the modified state is

$$\sum_{k=1}^{m} p_k U|\psi_k\rangle\langle\psi_k|U^\dagger = U\left(\sum_{k=1}^{m} p_k\, |\psi_k\rangle\langle\psi_k|\right)U^\dagger = U\rho U^\dagger.$$

Effect of a measurement on a density matrix:

measuring state ρ with respect to the basis $|\varphi_1\rangle$, $|\varphi_2\rangle$,…, $|\varphi_d\rangle$,
still yields the k^{th} outcome with probability $\langle\varphi_k|\rho|\varphi_k\rangle$.

Why?

Recap: density matrices

Quantum operations in terms of density matrices:

- Applying U to ρ gives $U\rho U^\dagger$
- Measuring state ρ with respect to the basis $|\varphi_1\rangle$, $|\varphi_2\rangle,\ldots,$ $|\varphi_d\rangle$ yields: k^{th} outcome with probability $\langle\varphi_k|\rho|\varphi_k\rangle$ —and causes the state to collapse to $|\varphi_k\rangle\langle\varphi_k|$.

Since these are expressible in terms of density matrices alone (independent of any specific probabilistic mixtures), states with identical density matrices are ***operationally indistinguishable***

Return to state distinguishing problems …

State distinguishing problems (1)

The ***density matrix*** of the mixed state $((|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), \ldots (|\psi_d\rangle, p_m))$ is $\rho = \sum_{k=1}^{m} p_k |\psi_k\rangle\langle\psi_k|$

Examples (from earlier in lecture):

1. & 2. $|0\rangle + |1\rangle$ and $-|0\rangle - |1\rangle$ both have $\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

3. $\begin{cases} |0\rangle \text{ with prob. } ½ \\ |1\rangle \text{ with prob. } ½ \end{cases}$

4. $\begin{cases} |0\rangle + |1\rangle \text{ with prob. } ½ \\ |0\rangle - |1\rangle \text{ with prob. } ½ \end{cases}$

6. $\begin{cases} |0\rangle \text{ with prob. } ¼ \\ |1\rangle \text{ with prob. } ¼ \\ |0\rangle + |1\rangle \text{ with prob. } ¼ \\ |0\rangle - |1\rangle \text{ with prob. } ¼ \end{cases}$

(3., 4. and 6. all have) $\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

State distinguishing problems (2)

Examples (continued):

5. $\begin{cases} |0\rangle & \text{with prob. } ½ \\ |0\rangle + |1\rangle & \text{with prob. } ½ \end{cases}$

has $\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$

7. The first qubit of $|01\rangle - |10\rangle$ **...? (later)**

Characterizing density matrices

$$\rho = \sum_{k=1}^{m} p_k |\psi_k\rangle\langle\psi_k|$$

Three properties of ρ:

- $\mathrm{Tr}\rho = 1$ $\left(\mathrm{Tr}M = M_{11} + M_{22} + \cdots + M_{dd}\right)$
- $\rho = \rho^\dagger$ (i.e. ρ is *Hermitian)*
- $\langle\varphi|\rho|\varphi\rangle \geq 0$, for all states $|\varphi\rangle$ (i.e. ρ is *positive semidefinite*)

Moreover, for ***any*** matrix ρ satisfying the above properties, there exists a probabilistic mixture whose density matrix is ρ

How do we show this?

Taxonomy of various normal matrices

Normal matrices

Definition: A matrix M is ***normal*** if $M^\dagger M = MM^\dagger$

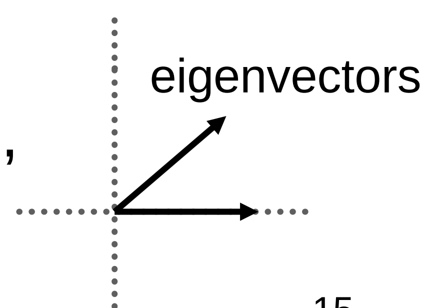
Theorem: M is normal iff there exists a unitary U such that $M = U^\dagger D U$, where D is diagonal (i.e. unitarily diagonalizable)

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}$$

Examples of ***ab***normal matrices:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not even diagonalizable

$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable, but not unitarily



Unitary and Hermitian matrices

Normal:

$$M = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}$$

with respect to ***some*** orthonormal basis

Unitary: $M^\dagger M = I$, which implies $|\lambda_k|^2 = 1$, for all k.

Hermitian: $M = M^\dagger$ which implies $\lambda_k \in \mathbb{R}$ for all k.

Question: Which matrices are both unitary ***and*** Hermitian?

Answer: Reflections ($\lambda_k \in \{+1, -1\}$ for all k).

Positive semidefinite

Positive semidefinite: Hermitian and $\lambda_k \geq 0$ for all k.

Theorem: M is positive semidefinite iff M is Hermitian and, for all $|\varphi\rangle$, $\langle\varphi|M|\varphi\rangle \geq 0$.

(**Positive *definite***: $\lambda_k > 0$ for all k)

Projections and density matrices

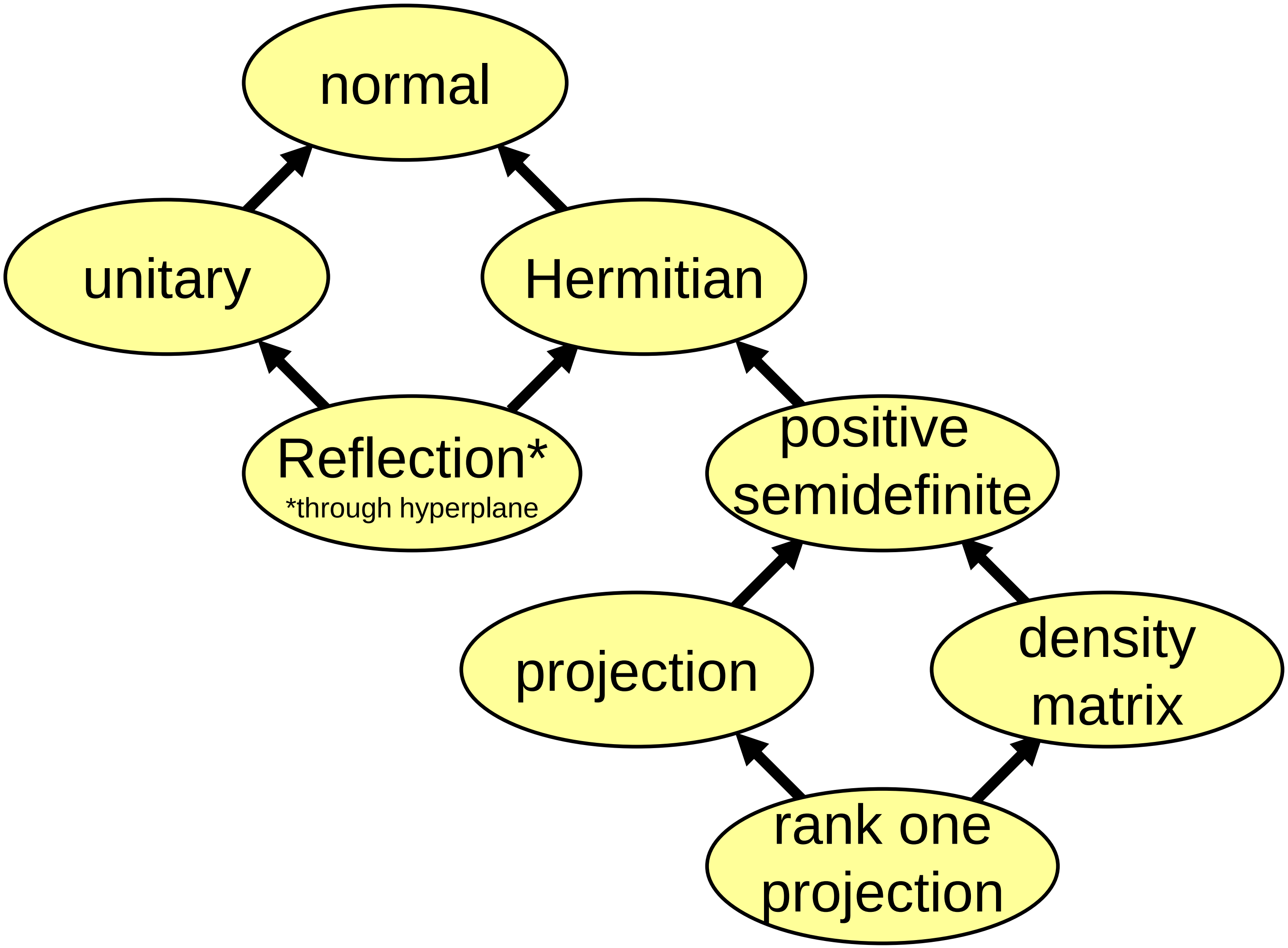
Projector: Hermitian and $M^2 = M$, which implies that M is positive semidefinite and $\lambda_k \in \{0,1\}$, for all k.

Density matrix: positive semidefinite and $\mathrm{Tr}\, M = 1$, so $\sum_{k=1}^{d} \lambda_k = 1$

Question: which matrices are both projections ***and*** density matrices?

Answer: rank-1 projections ($\lambda_k = 1$ if $k = j$, otherwise $\lambda_k = 0$)

Taxonomy of normal matrices



Bloch sphere for qubits

Bloch sphere for qubits (1)

Consider the set of all 2×2 density matrices ρ

They have a nice representation in terms of the ***Pauli matrices***

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that these matrices, combined with I, form a ***basis*** for the vector space of all 2×2 matrices.

We will express density matrices ρ in this basis.

Note: Coefficient of I must be $\frac{1}{2}$, since X, Y and Z are traceless.

Bloch sphere for qubits (2)

We can express $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$.

First consider the case of pure states $|\psi\rangle\langle\psi|$, where, without loss of generality, $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle$ for $0 \le \theta \le \pi, 0 \le \phi < 2\pi$ (unique if $\theta \ne 0, \pi$).

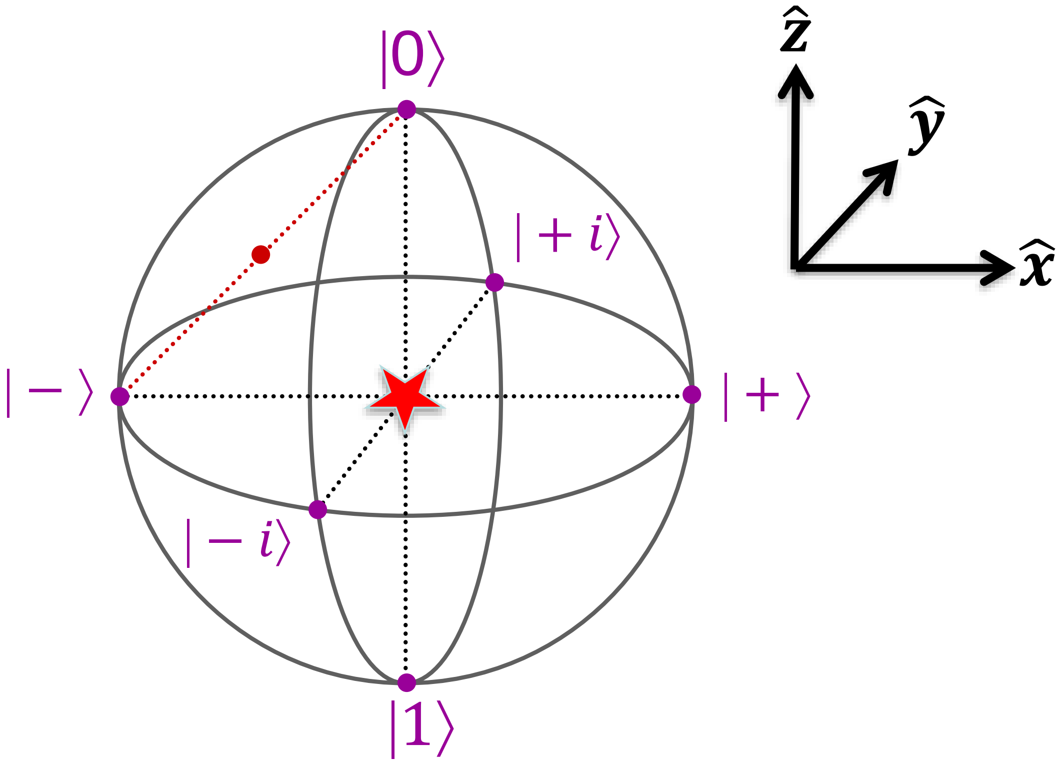
$$\rho = \begin{pmatrix} \cos^2(\theta/2) & e^{-i\phi}\cos(\theta/2)\sin(\theta/2) \\ e^{i\phi}\cos(\theta/2)\sin(\theta/2) & \sin^2(\theta/2) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+\cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & 1-\cos(\theta) \end{pmatrix}$$

Therefore $\boldsymbol{r} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$

These are ***polar coordinates*** of a unit vector $\boldsymbol{r} \in \mathbb{R}^3$

Bloch sphere for qubits (3)



$$|+\rangle = |0\rangle + |1\rangle$$
$$|-\rangle = |0\rangle - |1\rangle$$
$$|i\rangle = |0\rangle + i|1\rangle$$
$$|-i\rangle = |0\rangle - i|1\rangle$$

Note that ***orthogonal*** corresponds to ***antipodal*** here.

Pure states are on the surface ($|\boldsymbol{r}| = 1$), and mixed states are inside ($|\boldsymbol{r}| < 1$, being weighted averages of pure states).

The ***maximally mixed state*** $\rho = \frac{1}{2}I$ has $\boldsymbol{r} = 0$.