# Primitive ideals in quantum matrices

J. Bell

In this talk, we consider the ring of $m \times n$ quantum matrices $\mathcal{O}_q(M_{m,n})$.

The easiest way to visualize this algebra is that it is a complex algebra generated by elements $x_{i,j}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$. In addition to this we have a nonzero complex number $q$, which we'll assume is not a root of unity.

We can then put these elements $x_{i,j}$ in an $m \times n$ matrix and we declare that for each $2 \times 2$ submatrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have relations $ab = qba$, $cd = qdc$, $ac = qca$, $bd = qdb$, $bc = cb$, $ad - da = (q - q^{-1})bc$.

Notice that when $q = 1$, this just becomes the coordinate ring of the variety of $m \times n$ matrices. We can thus view this ring as a deformation of this coordinate ring. Indeed, it still has many properties in common with this coordinate ring.

- It has GK dimension $mn$.

- It is noetherian.

- It is an integral domain.

If we insist that $q$ not be a root of unity, then the ring will not satisfy a polynomial identity—unless $m = n = 1$ (but then what's the point, right?)

In this case, $H := (\mathbb{C}^*)^{m+n}$ surjects onto a subgroup of the automorphism group of $\mathcal{O}_q(M_{m,n})$ as follows. To an element

$$(\lambda_1, \ldots, \lambda_m; \gamma_1, \ldots, \gamma_n)$$

we associate the automorphism which sends $x_{i,j}$ to $\lambda_i \gamma_j x_{i,j}$.

This map is not injective as

$$(\lambda, \ldots, \lambda; \lambda^{-1}, \ldots, \lambda^{-1})$$

gives the identity automorphism.

## GOODEARL-LETZTER STRATIFICATION THEORY

Several results have been obtained regarding the primitive ideals of $\mathcal{O}_q(M_{m,n})$. It is known for instance that this algebra behaves roughly speaking as enveloping algebras of finite-dimensional complex nilpotent Lie algebras.

From work of Goodearl and Letzter it is known that the Dixmier-Moeglin equivalence holds for these algebras.

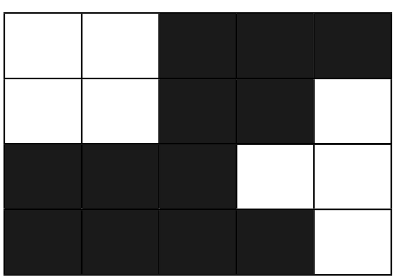
It follows from the work of Goodearl and Letzter that the number of prime ideals of $\mathcal{O}_q(M_{m,n})$ invariant under the action of this torus $H$ is finite.

Because of this, the prime spectrum of $\mathcal{O}_q(M_{m,n})$ admits a stratification into finitely many $H$-strata.

Each $H$-stratum is defined by a unique $H$-invariant prime ideal—that is minimal in its $H$-stratum—and is homeomorphic to the scheme of irreducible subvarieties of a torus. Moreover the primitive ideals correspond to those primes that are maximal in their $H$-strata.

Enumeration of the prime ideals in quantum matrices was done by Cauchon.

We say that an $m \times n$ grid is a *Cauchon diagram* if every square is coloured either black or white and whenever a square is coloured black, either every square in the same row that is to its left is also black or every square in the same column that is above it is also black.



Cauchon diagrams are named for their discoverer, who showed that there is a natural bijection between the collection of $m \times n$ Cauchon diagrams and the $H$-primes in the $\mathcal{O}_q(M_{m\times n})$

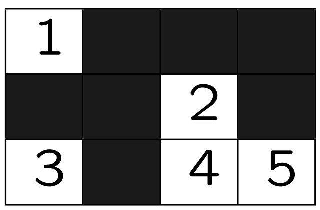
But how do we detect primitivity?

Let $C$ be an $m \times n$ labelled diagram with $k$ white squares labelled $\{1, 2, \ldots, k\}$. We define the *skew-adjacency matrix*, $M(C)$, of $C$ to be the $k \times k$ matrix whose $(i, j)$ entry is:

1. 1 if the square labelled $i$ is strictly to the left and in the same row as the square labelled $j$ or is strictly above and in the same column as the square labelled $j$;

2. $-1$ if the square labelled $i$ is strictly to the right and in the same row as the square labelled $j$ or is strictly below and in the same column as the square labelled $j$;

3. 0 otherwise.

$C:$  $\mapsto \; M(C) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix}$



The following theorem shows that primitivity is equivalent to invertibility of the skew-adjacency matrix of the corresponding diagram.

**THEOREM (B-Launois-Nguyen)**: Let $P$ be an $H$-invariant prime in $\mathcal{O}_q(M_{m,n})$ and let $C$ be its corresponding Cauchon diagram. Then $P$ is primitive if and only if $\det(M(C)) \neq 0$.

More generally, if $C$ is a Cauchon diagram corresponding to an $H$-prime $P$, then the nullity of the matrix $M(C)$ is precisely the transcendence degree of the centre of the quotient division algebra of $\mathcal{O}_q(M_{m,n})/P$ which is also the dimension of the corresponding $H$-stratum.

**THEOREM (B-Launois):** If $C$ is an $m \times n$ Cauchon diagram, then the nullity of the $M(C)$ is at most min$(m, n)$. More generally, every $H$-prime $P$ has the property that $\mathcal{O}_q(M_{m,n})/P$ has a primitive ideal whose height is at most min$(m, n)$.

- (Launois-Lenagan) The number of primitive $H$-invariant prime ideals in $\mathcal{O}_q(M_{1,n})$is

$$2^{n-1}.$$

- (B-Launois-Nguyen) The number of primitive $H$-invariant prime ideals in $\mathcal{O}_q(M_{2,n})$ is

$$(3^{n+1} - 2^{n+1} + (-1)^{n+1} + 2)/4.$$

- (B-Launois-Lutley) Let $n \geq 1$ be a natural number. Then the number of $3 \times n$ primitive $H$-primes in $\mathcal{O}_q(M_{3\times n})$ is given by

$$\frac{1}{8}\cdot(15\cdot 4^n - 18\cdot 3^n + 13\cdot 2^n - 6\cdot(-1)^n + 3\cdot(-2)^n).$$

More generally, B-Launois-Lutley showed that the if $m$ is fixed then the number of primitive $H$-primes in $\mathcal{O}_q(M_{m,n})$ satisfies a linear recurrence (as a function of $n$).

This was accomplished by showing that the determinant of the matrix $M(C)$ is either a power of 4 or is 0 (this was done by showing that the the determinant can be expressed in terms of a quadratic form mod 2).

Thus we can consider everything mod 3 when we look at the determinant. This finiteness allows us to obtain a linear recurrence.

Recently, we have begun looking at primitivity in terms of restricted permutations. (I should point out that Launois has told me that Yakimov has a very general result about primitivity of $H$-prime ideals, in this form, so I will only discuss enumeration results.)

The basic approach is to look at the kernel of $M(C)$ and show that there is a natural bijection between the kernel of this matrix and the kernel of a sum of two $(m+n) \times (m+n)$ permutation matrices. As a result finding the dimension of the kernel amounts to finding the number of cycles of even length in the disjoint cycle decomposition of a certain permutation.

We'll let $P(m,n,k)$ denote the number of $H$-primes $P$ in $m \times n$ quantum matrices for which the quotient division algebra of $\mathcal{O}_q(M_{m,n})/P$ has transcendence degree $k$ over $\mathbb{C}$.

We are able to find the generating series

$$\sum_{m,n,k} P(m,n,k)x^m y^n t^k/m!n!$$

**THEOREM (B-Kasteels-Launois:**

$$\sum_{m,n,k} P(m,n,k) x^m y^n t^k / m! n!$$

has the following closed form:

$$e^{x+y} \left( \frac{1-(1-e^{-x})(1-e^{-y})}{1-(1-e^{x})(1-e^{y})} \right)^{1/2}$$

$$\times \left( \frac{1}{(1-(1-e^{x})(1-e^{y}))(1-(1-e^{-x})(1-e^{-y}))} \right)^{t/2}$$

We note that if we set $t = 1$, we get the formula for the e.g.f. for $H$-primes, which was proved by Cauchon:

$$\sum_{m,n,k} P(m,n)x^m y^n / m!n!$$

is equal to

$$e^{x+y}\left(\frac{1}{(1-(1-e^x)(1-e^y))}\right)$$

Let Prim$(m, n)$ denote the number of primitive $H$-primes in $\mathcal{O}_q(M_{m,n})$.

Since an $H$-prime $P$ is primitive if and only if the quotient division algebra of $\mathcal{O}_q(M_{m,n})/P$ has transcendence degree 0 over $\mathbb{C}$ (i.e., it is the complex numbers), we can set $t = 0$ and obtain the formula:

$$\sum_{m,n,k} \text{Prim}(m,n) x^m y^n / m! n!$$

has the following closed form:

$$e^{x+y} \left( \frac{1 - (1 - e^{-x})(1 - e^{-y})}{1 - (1 - e^{x})(1 - e^{y})} \right)^{1/2}$$

Consequences

Using this generating function, we can prove many results conjectured by Launois and me.

If $m$ is fixed then the proportion of $H$-primes in $\mathcal{O}_q(M_{m,n})$ for which the $H$-stratum is $k$ dimension tends to

$$2\binom{2m}{m+k}/4^m$$

if $k > 0$ and to

$$\binom{2m}{m}/4^m$$

if $k = 0$ as $n \to \infty$.

If $m$ is fixed then there are constants $c_j$ for $1 - m \leq j \leq m + 1$ such that the number of primitive $H$-primes in $\mathcal{O}_q(M_{m,n})$ is of the form

$$\sum_{j=2-m}^{m+1} c_j j^n$$

for all $n$.

In fact, we can get a closed form for these constants $c_j$.