Subfields of division algebras

J. Bell
In this talk, we consider domains which are finitely generated over an algebraically closed field $F$.

Given such a domain $A$, we know that if $A$ does not contain a copy of the free algebra on two generators then $A$ has a quotient division algebra $Q(A)$ formed by inverting the nonzero elements of $A$.

We are interested in the case when $A$ has finite GK dimension.
GK dimension is a measure of the growth of an algebra. Specifically, we take a finite dimensional subspace $V$ of $A$ that contains 1 and generates $A$ as an algebra and we look at how

$$\dim(V^n)$$

grows as a function of $n$. If it grows like a polynomial of degree $d$, then we say that $A$ has $GK$ dimension $d$. 
Much is known:

- If $A$ has GK dimension 0 then $A = F$.

- If $A$ has GK dimension 1, then $A$ is commutative.

GK 2 is still mysterious
A natural question is to ask if we can classify the quotient division algebras of domains of GK dimension 2.

This is very difficult, but some progress has been made.

**THEOREM:** (Artin-Stafford) If $A$ is a domain of GK dimension 2 which is finitely generated over a field $F$ then $Q(A) = K(x; \sigma)$, where $K$ is a finitely generated field of transcendence degree 1 over $F$.

One of the important questions Artin and Stafford posed was whether a graded domain of GK dimension strictly between 2 and 3 could exist?
Smoktunowicz showed that it could not. Her result could be interpreted as follows: If $D = Q(A)$ is a division algebra over an algebraically closed field $K$ and

$$K \subseteq F \subseteq E \subseteq D$$

are division algebras, each one infinite dimensional as a vector space over the preceding one and $D = E(x; \sigma)$, then $A$ has GK dimension at least 3.

Note that if $A$ is graded, then $A$ has a division algebra of the form $E(x; \sigma)$, where $E$ is the degree 0 piece of the homogeneous graded quotient ring.
We show the following result.

**THEOREM:** Let $D = Q(A)$ be a quotient division algebra and suppose we have a chain

$$K = D_0 \subseteq D_1 \subseteq D_2 \subseteq D_3$$

in $D$ with each $D_i$ infinite dimensional as a left $D_{i-1}$-vector space. Then $A$ has GK dimension at least 3.
As a corollary we get the following result.

**THEOREM:** Let $A$ be a domain of GK dimension 2 which is finitely generated domain over an algebraically closed field $K$. If $D$ is a division subalgebra of $Q(A)$ then either $D$ is a field or $Q(A)$ is finite dimensional as a left $D$-vector space.

Why? Pick $x \in D \setminus K$ and consider the chain

$$K \subseteq K(x) \subseteq D \subseteq Q(A).$$

Use Tsen’s theorem!
Let's look at division algebras of algebras of higher GK.
One of the important problems for division algebras: find invariants for quotient division algebras which generalize many of the properties of transcendence degree for quotient fields of commutative domains.

In this direction, Zhang has done a lot of work. He has given a few such invariants. One important invariant is *Lower transcendence degree*. Alas, I don’t have time to define it!

Zhang showed this degree has the following properties for a division algebra $D$:

- $\text{Ld}(K) = \text{Transcendence degree of } K$;
- $\text{Ld}(K) \leq \text{Ld}(D)$ for a subfield $K$ of $D$;
- If $D = Q(A)$ then $\text{Ld}(D) = \text{GKdim}(A)$. 


In particular, Zhang’s results show that if $A$ has GK dimension $d$, then any subfield of $\mathbb{Q}(A)$ has transcendence degree at most $d$. 
We consider the question of subfields of a division algebra. One thing to note is that the only known examples where $\mathbb{Q}(A)$ has a subfield of transcendence degree equal to the GK dimension of $A$ is when $\mathbb{Q}(A)$ is finite dimensional over its centre.

Is this the only case where such a situation can occur?

To look at this problem we use techniques from Zhang and Smoktunowicz
Zhang was able to show that if $D = F(x; \sigma)$ then $\text{Ld}(D) \geq 1 + \text{transcendence degree of } F$. (He did something much stronger than this, in fact—but a section is devoted to skew polynomial extensions.)

We’d like to try to replace the hypothesis that $D = F(x; \sigma)$ by simply saying that $D$ is infinite dimensional over $F$ as a left $F$-vector space and $D$ is finitely generated as a division algebra.
Unfortunately, there is a key difficulty. It is difficult to distinguish between being finite dimensional over a field and being algebraic over the field.

**DEFINITION:** Let $D$ be a division algebra and let $K$ be a subfield of $D$. We say that $D$ is left algebraic over $K$ if for every $x \in D$ we have an equation of the form

$$x^n + \beta_{n-1}x^{n-1} + \cdots + \beta_0 = 0$$

for some $\beta_i \in K$.

If we can avoid this, then we are fine.

**THEOREM:** Let $A$ be a domain of GK dimension $d$ which is finitely generated over a field $K$. If $F$ is a subfield of $Q(A)$ and $Q(A)$ is not left algebraic over $F$ then the transcendence degree of $F$ is at most $d - 1$. 
But when can we know for sure that $Q(A)$ is not left algebraic over a subfield $F$.

Answer: The Poincaré-Birkhoff-Witt theorem tells us this.

An algebra has a PBW basis if there are $x_1, \ldots, x_d \in A$ such that

$$\{x_1^{i_1} \cdots x_d^{i_d} \mid i_1, \ldots, i_d \geq 0\}$$

is a basis for $A$.

**Theorem:** If $A$ has a PBW basis, then $Q(A)$ is left algebraic over a field $F$ if and only if $Q(A)$ is finite dimensional over $F$. 

15
QUESTION: If $Q(A)$ is left algebraic over a subfield $K$ is it necessarily algebraic over its centre?
One last question: When are the subfields finitely generated field extensions of $K$?
This is difficult in general. One approach is to use the following fact.

**FACT:** $F$ is a finitely generated field extension of $K$ iff $F \otimes_K F$ is Noetherian.
Using this fact one can show that if $K$ is uncountable in addition to being algebraically closed and $A$ is a finitely generated Noetherian (!!!) domain over $K$ then all subfields of $\mathbb{Q}(A)$ are finitely generated field extensions of $K$. 