Subfields of division algebras

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In this talk, we consider domains which are finitely generated over an algebraically closed field F.

Given such a domain A, we know that if A does not contain a copy of the free algebra on two generators then A has a quotient division algebra $Q(A)$ formed by inverting the nonzero elements of A.

We are interested in the case when A has finite GK dimension.

GK dimension is a measure of the growth of an algebra. Specifically, we take a finite dimensional subspace V of A that contains 1 and generates A as an algebra and we look at how

$$\dim(V^n)$$

grows as a function of n. If it grows like a polynomial of degree d, then we say that A has *GK dimension* d.

Much is known:

- If A has GK dimension 0 then $A = F$.

- If A has GK dimension 1, then A is commutative.

GK 2 is still mysterious

A natural question is to ask if we can classify the quotient division algebras of domains of GK dimension 2.

This is very difficult, but some progress has been made.

THEOREM: (Artin-Stafford) If A is a domain of GK dimension 2 which is finitely generated over a field F then $Q(A) = K(x;\sigma)$, where K is a finitely generated field of transcendence degree 1 over F.

One of the important questions Artin and Stafford posed was whether a graded domain of GK dimension strictly between 2 and 3 could exist?

Smoktunowicz showed that it could not. Her result could be interpreted as follows: If $D = Q(A)$ is a division algebra over an algebraically closed field K and

$K \subseteq F \subseteq E \subseteq D$ are division algebras, each one infinite dimensional as a vector space over the preceding one and $D = E(x;\sigma)$, then A has GK dimension at least 3.

Note that if A is graded, then A has a division algebra of the form $E(x;\sigma)$, where E is the degree 0 piece of the homogeneous graded quotient ring.

We show the following result.

THEOREM: Let $D = Q(A)$ be a quotient division algebra and suppose we have a chain

$$K = D_0 \subseteq D_1 \subseteq D_2 \subseteq D_3$$

in D with each D_i infinite dimensional as a left D_{i-1}-vector space. Then A has GK dimension at least 3.

As a corollary we get the following result.

THEOREM: Let A be a domain of GK dimension 2 which is finitely generated domain over an algebraically closed field K. If D is a division subalgebra of $Q(A)$ then either D is a field or $Q(A)$ is finite dimensional as a left D-vector space.

Why? Pick $x \in D \setminus K$ and consider the chain

$$K \subseteq K(x) \subseteq D \subseteq Q(A).$$

Use Tsen's theorem!

Let's look at division algebras of algebras of higher GK.

One of the important problems for division algebras: find invariants for quotient division algebras which generalize many of the properties of transcendence degree for quotient fields of commutative domains.

In this direction, Zhang has done a lot of work. He has given a few such invariants. One important invariant is *Lower transcendence degree*. Alas, I don't have time to define it!

Zhang showed this degree has the following properties for a division algebra D:

- Ld(K) = Transcendence degree of K;
- Ld(K) $\leq$Ld(D) for a subfield K of D;
- If $D = Q(A)$ then Ld(D) = GKdim(A).

In particular, Zhang's results show that if A has GK dimension d, then any subfield of $Q(A)$ has transcendence degree at most d.

We consider the question of subfields of a division algebra. One thing to note is that the only known examples where $Q(A)$ has a subfield of transcendence degree equal to the GK dimension of A is when $Q(A)$ is finite dimensional over its centre.

Is this the only case where such a situation can occur?

To look at this problem we use techniques from Zhang and Smoktunowicz

Zhang was able to show that if $D = F(x;\sigma)$ then $\mathrm{Ld}(D) \geq 1+$ transcendence degree of F. (He did something much stronger than this, in fact–but a section is devoted to skew polynomial extensions.)

We'd like to try to replace the hypothesis that $D = F(x;\sigma)$ by simply saying that D is infinite dimensional over F as a left F-vector space and D is finitely generated as a division algebra.

Unfortunately, there is a key difficulty. It is difficult to distinguish between being finite dimensional over a field and being algebraic over the field.

DEFINITION: Let D be a division algebra and let K be a subfield of D. We say that D is left algebraic over K if for every $x \in D$ we have an equation of the form

$$x^n + \beta_{n-1}x^{n-1} + \cdots + \beta_0 = 0$$

for some $\beta_i \in K$.

If we can avoid this, then we are fine.

THEOREM: Let A be a domain of GK dimension d which is finitely generated over a field K. If F is a subfield of $Q(A)$ and $Q(A)$ is not left algebraic over F then the transcendence degree of F is at most $d-1$.

But when can we know for sure that $Q(A)$ is not left algebraic over a subfield F.

Answer: The Poincaré-Birkhoff-Witt theorem tells us this.

An algebra has a PBW basis if there are $x_1, \ldots, x_d \in A$ such that

$$\{x_1^{i_1} \cdots x_d^{i_d} \mid i_1, \ldots, i_d \geq 0\}$$

is a basis for A.

THEOREM: If A has a PBW basis, then $Q(A)$ is left algebraic over a field F if and only if $Q(A)$ is finite dimensional over F.

QUESTION: If $Q(A)$ is left algebraic over a subfield K is it necessarily algebraic over its centre?

One last question: When are the subfields finitely generated field extensions of K?

This is difficult in general. One approach is to use the following fact.

FACT: F is a finitely generated field extension of K iff $F \otimes_K F$ is Noetherian.

Using this fact one can show that if K is uncountable in addition to being algebraically closed and A is a finitely generated Noetherian (!!!) domain over K then all subfields of $Q(A)$ are finitely generated field extensions of K.