Critical density, automorphisms, and Harm Derksen

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Let X be a topological space and let $\mathcal{S}$ be an infinite subset of X.

We say that $\mathcal{S}$ is *critically dense* if every infinite subset of $\mathcal{S}$ is dense in X.

Equivalently, $\mathcal{S} \cap Y$ is a finite set for every proper closed subset Y of X.

This notion was introduced by Artin-Small-Zhang in their paper on generic flatness.

Rogalski showed that the critical density property was closely linked with the Noetherian property for twisted homogeneous coordinate rings.

Brief description of Rogalski's constructions:

E.g.,

Let σ be an automorphism of $\mathbb{P}^d$. We form the *twisted homogeneous coordinate ring* of $\mathbb{P}^d$ with respect to σ:

$$\bigoplus_{i=0}^{\infty} \mathsf{H}^0(\mathbb{P}^d, \mathcal{O}(1) \otimes \mathcal{O}(1)^{\sigma} \otimes \cdots \otimes \mathcal{O}(1)^{\sigma^{i-1}}).$$

Zhang's description of this ring:

The n'th homogeneous part can be identified with the space of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_d]$ of degree n.

Multiplication, however, is twisted in this ring by the formula

$$f \star g \;=\; f\phi^m(g),$$

where f is homogeneous of degree m and ϕ is the automorphism of $\mathbb{C}[x_0, \ldots, x_d]$ induced by σ.

Notice that to a point $\mathbf{c} \in \mathbb{P}^d$, we can associate a codimension 1 subspace of $\mathbf{H}^0(\mathbb{P}^d, \mathcal{O}(1))$.

Define $R(\sigma, \mathbf{c})$ to be the subring of the twisted homogeneous coordinate ring generated by this codimension 1 subspace.

THEOREM: $R(\sigma, \mathbf{c})$ is Noetherian if and only if

$$\{\sigma^n(\mathbf{c}) \mid n \in \mathbb{Z}\}$$

is critically dense in $\mathbb{P}^d$.

ROGALSKI'S QUESION: Can "critically dense" be replaced by "dense" in the theorem; i.e., is density equivalent to critical density for sets of the form

$$\{\sigma^n(\mathbf{c}) \mid n \in \mathbb{Z}\},$$

where $\mathbf{c}$ is a point in a quasi-projective variety?

The answer to Rogalski's question is 'no' if we are working in positive characteristic, as the following example due to Lech shows.

Let $X = \mathbb{A}^2$ over the field $\mathbb{F}_p(t)$. Let σ be the automorphism given by

$$\sigma(x, y) \ = \ (tx, (1+t)y).$$

Take $\mathbf{c} = (1, 1)$.

Notice that

$$\sigma^{p^n}(1, 1) \ = \ (t^{p^n}, (1+t)^{p^n}) = (t^{p^n}, 1 + t^{p^n}).$$

Hence the orbit of $(1, 1)$ under σ is in the variety $Z(y - x - 1)$ infinitely often and so the orbit is not critically dense. It is not hard to show, however, that the orbit must be dense.

What happens in characteristic 0?

The characteristic 0 case is still open, but can be shown for many classes of varieties using p-adic methods.

- $\mathbb{P}^d$ and more generally any projective variety in which σ fixes an ample invertible sheaf (Rogalski)
- Fano varieties
- Abelian varieties (for translations, at least) (Rogalski)
- many surfaces (but still not all projective surfaces) Rogalski

- curves (of course)
- Toric varieties in which σ is a toric automorphism
- All affine varieties

Rogalski's work:

Rogalski relied heavily on the theorem of Srinivas and Cutkosky

THEOREM (C-S) Let F be an algebraically closed field of char. 0 and let G be an algebraic group defined over F. Suppose that a cyclic subgroup H is dense in G. Then H is critically dense.

With this theorem he could prove his conjecture for projective space and abelian varieties in the case that σ is a translation.

Here is an over-simplified picture of how the proof works for affine d-space.

- Show that it is no loss of generality to assume that you are working over a p-adic field $\mathbb{Q}_p$.

- Show that the automorphism $\sigma : \mathbb{A}^d \longrightarrow \mathbb{A}^d$ has the property that there exist analytic maps $f_1, \ldots, f_d : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ such that

$$\sigma^{an+b}(\mathbf{c}) = (f_1(n), \ldots, f_d(n))$$

for some a and b

- Argue that if $\sigma^{an+b}(\mathbf{c})$ is in $Z(P(x_1, \ldots, x_d))$ for infinitely many n, then $P(f_1(x), \ldots, f_d(x))$ is an analytic function with infinitely many zeros in the unit disc.

- Conclude that it is identically zero and hence $\sigma^{an+b}(\mathbf{c})$ must be in $Z(P)$ for all n.

Using this approache we have the following results:

THEOREM: Let σ be an automorphism of an affine variety over a field of characteristic 0. Then $\{\sigma^n(p) \mid n \in \mathbb{Z}\}$ is dense if and only if it is critically dense.

THEOREM: Let σ be an automorphism of an affine variety over a field of characteristic 0 and let Y be a subvariety of $\mathbb{A}^d$. Then $\{n \mid \sigma^n(p) \in Y\}$ is a finite union of 2-way arithmetic progressions possibly augmented by a finite set.

This can be seen as a generalization of the Skolem-Mahler-Lech therorem.

THEOREM (S-M-L) Let $f(z)$ be the power series expansion of a rational function. Then the set of n such that the coefficient of z^n in $f(z)$ is 0 is a finite union of one-way arithmetic progressions possibly augmented by a finite set.

The reason for this is that any rational function $f(z)$ can be realized as

$$\sum_{i=0}^{\infty} \mathbf{w}^{\top} A^n \mathbf{v} z^n.$$

Up to a finite number of coefficients, this series will agree with one in which A is an invertible matrix. Thus it is no loss of generality to assume that A is invertible. Notice that A can be thought of as giving a "linear" automorphism of $\mathbb{A}^d$, $\mathbf{v}$ can be thought of as a point in $\mathbb{A}^d$. Then saying that $\mathbf{w}^T A^n \mathbf{v} = 0$ is the same as saying that $A^n \mathbf{v}$ is in the hyperplane given by $\mathbf{w}^T \mathbf{x} = 0$.

Thus it makes sense to give a stronger conjecture:

STRONG ROGALSKI CONJECTURE: Let X be a q.p. variety over a field of characteristic 0 with automorphism σ. If $x \in X$ and Y is a subvariety of X Then $\{n \mid \sigma^n(x) \in Y\}$ is a finite union of 2-way arithmetic progressions possibly augmented by a finite set.

This conjecture is true in all the cases Rogalski's conjecture is known to be true, except for abelian varieties.

Can this conjecture be done for surfaces?

Look at the Kodaira dimension of a surface. In the case that the Kodaira dimension is 2, the automorphism group is finite. In the case that the Kodaira dimension is 0 or 1, our surface will have a unique minimal model and it is therefore sufficient to work with this surface. Nonnegative Kodaira dimension seems like is should be doable. The rational/ruled case might be harder, but there is also a lot of nice structure that can be exploited.

Another approach. Is there some sort of theorem which says give a projective variety X with automorphism σ, then there is some "nice" projective variety Y such that X embeds in Y and σ extends to an automorphism of Y.

What happens in positive characteristic?

As the example of Lech shows, the picture is more complicated. Nevertheless, the great Dutch problem solver comes to our rescue and gives a partial answer to the problem. To explain, let's first look at some examples of *automatic sequences*.

A sequence is p-*automatic* if its n'th term is generated by a finite state machine accepting the digits of n in base p as input.

Example: Thue-Morse sequence

$$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$$

is 2-automatic.

A subset of $\mathbb{N}$ is p-automatic if the sequence $f(n)$ defined to be 1 if n is in the set and 0 if it is not is a p-automatic sequence.

Harm Derksen has shown the following amazing fact:

THEOREM: Let K be a field of char. $p > 0$ and let X be a subvariety of $\mathbb{P}^n(K)$ and let $c \in \mathbb{P}^n$. Then the set of n such that $\sigma(n) \in X$ is a p-automatic set

Thus to check critical density it is enough to check arithmetic progressions along with other sets defined in terms of base p expansions.

In light of Derksen's result, the following conjecture seems reasonable:

CONJECTURE: Let K be a field of char. $p > 0$ and let X be a q.p. variety over K. If $x \in X$ and Y is a subvariety of X, then

$$\{n \in \mathbb{N} \mid \sigma^n(x) \in Y\}$$

is p-automatic.