Hilbert series of affine prime PI rings

GK DIMENSION

Given a finitely generated k-algebra A and a finite dimensional k-vector space V such that

- $1_A \in V$
- k[V]=A

we define

$$\mathrm{GKdim}(A) = \limsup_{n\to\infty} \log(\dim V^n)/\log(n).$$

For example, if $A = k[x_1, \ldots, x_d]$, we can take $V = k + kx_1 + \cdots + kx_d$. We have

$$\text{dim } V^n = \binom{n+d}{d} \sim n^d/d!.$$

Therefore

$$\text{GKdim}(A) = \lim_{n\to\infty} \log(n^d/d!)/\log(n) = d.$$

More generally, the GK dimension of an affine commutative k-algebra is the same as its Krull dimension.

- If A is affine, then A has GK dimension 0 if and only if it is finite dimensional.

- If A is affine and has GK dimension 1, then A is PI.

- There are no algebras of GK dimension strictly between 0 and 1 or strictly between 1 and 2

- For any real number $\alpha \geq 2$, there exists an algebra of GK dimension α.

Significance of Hilbert Series:

Hilbert Series tell us about the "growth" of an algebra. If the Hilbert series $H(t)$ of an algebra A is rational, then we have either:

- A has exponential growth; or
- A has integer GK dimension–in this case, it is given by the order of the pole at $t = 1$ in $H(t)$.

QUESTION: Does an affine PI algebra necessarily have a rational Hilbert series (with respect to some standard filtration)?

ANSWER: NO! Constructions of Warfield show this is not the case. Let

$$I = (y)^3 + (yx^i y \mid i \text{ not a perfect square}).$$

Then $A = F\{x, y\}/I$ has GK dimension 2.5. Moreover the image of $(y)^3$ is (0) in A. Thus A satisfies $[t_1, t_2]^3$.

Problem: Warfield's construction shows that the nilpotent radical of an affine PI algebra is not in general well-behaved. Thus we restrict our consideration to the prime affine PI case.

Even in this case, problems can arise. Stafford has shown that there exist PI algebras which have rational Hilbert series with respect to some standard filtration, but which are not rational with respect to another. Thus some care has to be exercised in finding the vector space with which to filter the algebra.

- In computing the Hilbert series of an algebra, *Gröbner bases* are often quite useful.

- The main idea is that one can use Gröbner bases to find a monomial algebra with the same Hilbert series as the given algebra.

If

$$A = F\{x_1, \ldots, x_n\}/I,$$

we can put an ordering on $x_1, \ldots, x_n$ by declaring

$$x_1 < x_2 < \cdots < x_n.$$

Once this is done, we can order the collection of words in $x_1, \ldots, x_n$ with a *degree lexicographic ordering*. This says that $W_1 < W_2$ if W_2 is either longer than W_1 or if they have the same length but W_2 is larger than W_1 in the lexicographic ordering.

Given an element $a \in F\{x_1, \ldots, x_n\}$, we let

$$\text{in}(a)$$

denote the largest (or initial) monomial which appears in the expression for a. Given an ideal I, we let

$$\text{in}(I)$$

denote the ideal generated by the $\text{in}(a)$ for $a \in I$.

Theorem: The algebras $A = F\{x_1, \ldots, x_n\}/I$ and $B = F\{x_1, \ldots, x_n\}/\text{in}(I)$ have the same Hilbert series.

Theorem: (Anick, Ufnarovskii) Let A be a finitely presented monomial algebra; i.e.,

$$A = F\{x_1, \ldots, x_n\}/(W_1, \ldots, W_m).$$

Then A has a rational Hilbert series. Moreover, if we let R denote the radius of convergence of the Hilbert series, we have the following cases:

- If $R < 1$, then A contains a copy of the free algebra on two generators;

- If $R = 1$, then A has integer GK dimension equal to the order of the pole at $t = 1$ in $H_A(t)$; or

- If $R > 1$, then $R = \infty$ and A is finite dimensional over F and $H_A(t)$ is a polynomial.

From this we see that if we can order our variables in some way so that the Gröbner basis algorithm terminates after a finite number of steps, then our algebra has a rational Hilbert series. There is a more general criterion which allows one to see if a monomial algebra has a rational Hilbert series. As an example, consider the algebra

$$F\{x, y\}/(yx^j y \mid j \geq 0).$$

Algebras with such an associated digraph are called *automata algebras*.

THEOREM: If A is an automata algebra, then A has a rational Hilbert series.

BACK TO PI RINGS!

The straightforward approach to showing that a finitely generated prime PI ring has a rational Hilbert series is the following.

- First take a central element $z \in A$ such that when z is inverted, the resulting ring A_z is a finite module over its center.

- By the Artin-Tate lemma the center of A_z is a finitely generated commutative ring and hence has rational Hilbert series.

- Use this Hilbert series and the fact that A_z is a finite module over its center to show that A_z has rational Hilbert series.

- Somehow "pullback" the information about A_z to A and show that it too has a rational Hilbert series.

How to get this to work: carefully pick the z which you are going to invert using Gröbner techniques. Once you have done this, you will have a lot of information about how it interacts with other elements of the algebra.

FACT: If A is an affine prime PI ring then $\mathrm{GKdim}(A) = d$ for some nonnegative integer d.

FACT: If A is an affine prime PI ring of GK dimension d, then

$$k[z_1, \ldots, z_d] \subseteq A.$$

Now what do we do? Notice that if we pick generators for our PI algebra A, say $x_1, \ldots, x_m$. Let V be the vector space spanned by these elements, 1, and by $z_1, \ldots, z_d$. Let W be the vector space spanned by $1, z_1, \ldots, z_d$. Then

$$\dim(W^n) = \binom{n+d}{d},$$

and

$$\dim(V^n) \leq Cn^d$$

for some $C > 0$.

WHAT IS THE SIGNIFICANCE OF THIS FACT?

Example: Consider a word u in $x_1, \dots, x_m$. As an example, look at $u = x_1x_2x_3$. Consider

$$x_1x_2x_3W^{n-3} + x_1x_2W^{n-2} + x_1W^{n-1} + W^n.$$

This space is contained in V^n.

If this set sum is direct, then it has dimension at least

$$\binom{n+d}{n} + \binom{n+d-1}{d} + \binom{n+d-2}{d} + \binom{n+d-3}{d}$$
$$\sim \; 4n^d/d! = \text{length}(u)n^d/d!.$$

Thus if u is long enough this sum cannot be direct. It follows that there is some N such that every word, u, of length N has an initial subword v such that there is a polynomial q_u with the property that

$$0 = vq_u(z_1, \ldots, z_d) + \text{lower lex. order terms.}$$

It follows that if we invert

$$z = \prod_{u \text{ of length } N} q_u,$$

then our algebra is a finite module over its center.

Having constructed this z, if we are careful in picking the right set of generators for our algebra, we see that it can be argued that A_z has a rational Hilbert series and that we can "pullback" and get that A has a rational Hilbert series.

CONJECTURES:

Is the result true for affine Noetherian PI rings (without the prime hypothesis)?

Are prime PI rings finitely presented?

THE END