Affinization

History:

In the 50s, Amitsur proved Koethe's conjecture for algebras over an uncountable field by using a short and clever counting argument. This same argument could be used to show that if k is a field and A is a k-algebra with $[A : k] < |k|$, then

- If A is primitive, $Z(A) = k$

- $J(A)$ is nil.

In particular an affine algebra over an uncountable field has nil Jacobson radical.

TWO NATURAL QUESTIONS:

- If A is an affine k-algebra (k not necessarily uncountable) is $J(A)$ nil?

- If, in addition, A is Noetherian, is $J(A)$ nilpotent?

The answer to the second question is not known.

The first question was answered by Beĭdar (1980), who showed that for countable fields, the Jacobson radical need not be nil. In 1981, Small gave a simpler more general construction. This construction is known as *affinization*.

Construction

- Let k be a field.

- Let R be a prime countably generated k-algebra.

Let

$$S = \begin{pmatrix} k + k\{x,y\}y & k\{x,y\} \\ k\{x,y\}y & k\{x,y\} \end{pmatrix}.$$

FACTS:

- S is finitely generated as a k-algebra

- $k + k\{x,y\}y$ is the countably generated free algebra on generators $\{x^i y \mid i \geq 0\}$.

There exists a surjective homomorphism

$$\Phi : k + k\{x, y\}y \to R.$$

There exists a prime ideal $Q \subseteq S$ such that

$$e_{1,1}Qe_{1,1} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}.$$

$A := S/Q$ is the *affinization* of R.

GK DIMENSION

Given a finitely generated k-algebra A and a finite dimensional k-vector space V such that

- $1_A \in V$
- k[V]=A

we define

$$\text{GKdim}(A) = \limsup_{n\to\infty} \log(\dim V^n)/\log(n).$$

For example, if $A = k[x_1, \ldots, x_d]$, we can take $V = k + kx_1 + \cdots + kx_d$. We have

$$\dim V^n = \binom{n+d}{d} \sim n^d/d!.$$

Therefore

$$\mathrm{GKdim}(A) = \lim_{n \to \infty} \log(n^d/d!)/\log(n) = d.$$

More generally, the GK dimension of an affine commutative k-algebra is the same as its Krull dimension.

If A is not affine, we define its GK dimension to be the supremum of the GK dimensions of its affine subalgebras.

- A has GK dimension 0 if and only if it is locally finite.

- If A has GK dimension 1, then A is PI.

- There are no algebras of GK dimension strictly between 0 and 1 or strictly between 1 and 2

- For any real number $\alpha \geq 2$, there exists an algebra of GK dimension α.

Affinization and GK dimension Recall we had a countably generated k-algebra and a surjection

$$\Phi : k + k\{x, y\}y \to R.$$

By carefully defining Φ, it can be shown that there is a prime ideal $Q \subseteq S$ with

$$\overline{e_{1,1}}(S/Q)\overline{e_{1,1}} \;\cong\; R$$

and

$$\mathrm{GKdim}(S/Q) \;=\; \mathrm{GKdim}(R) + 2.$$

The algebra A/Q never:

- is Noetherian or even Goldie;
- has a rational Hilbert series with respect to some standard filtration;
- is PI

The algebra A/Q is:

- prime;
- primitive if (and only if) R is primitive;
- centerless.

APPLICATIONS Let k be countable. Take $R = k[t]_{(t)}$. Then $\mathrm{GKdim}(R) = 1$. So we can construct an affine k-algebra of GK dimension 3 with non-nil Jacobson radical.

$$\overline{e_{1,1}} J(A) \overline{e_{1,1}} \cong J(R) = tk[t]_{(t)}.$$

What happens for affine algebras of GK dimension 2?

Q: (Goodearl) If $J(A)$ nil if A is an affine algebra of GK dimension 2? If, in addition, A is Noetherian, is $J(A)$ nilpotent?

Let R be a countably generated prime k-algebra of GK dimension 0 which is not primitive. For example, take

$$R = \left\{ \begin{pmatrix} C & 0 & \cdots \\ 0 & C & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} : C \text{ is upper} - \text{triangular} \right\}.$$

Then there exists an affine k-algebra of GK dimension 2 with

$$\overline{e_{1,1}} A \overline{e_{1,1}} \cong R.$$

Thus A is a prime affine algebra of GK dimension 2, but it is neither primitive nor PI.

NOTE: A does not have quadratic growth.

Q: (Small) If A is an affine algebra of GK dimension 2 and A is Noetherian, is A necessarily primitive or PI?

Other examples which can be constructed this way:

- A primitive ring of GK 3 (over a countable field) with center not equal to a field.
- An affine ring of GK 3 which does not satisfy the Nullstellensatz.
- An affine ring of GK 2 for which GK dimension is not finitely partitive.
- An affine ring of GK 2 with infinite classical Krull dimension.

Another modification

If we think of $k\{x, y\} = k[x] \star k[y]$, we can do the same construction with any affine algebra W.

Let $W\{y\} \;=\; W \star k[y]$. Let

$$S \;=\; \begin{pmatrix} k + W\{y\}y & W\{y\} \\ W\{y\}y & W\{y\} \end{pmatrix}.$$

As before, S is affine and $e_{1,1}Se_{1,1}$ is free on $\{w_i y\}$, where $\{w_i\}$ is a basis for W as a k-vector space.

Once again, given a countably generated prime k-algebra, R, we can find a prime ideal Q of S such that

$$\overline{e_{1,1}}(S/Q)\overline{e_{1,1}} \;\cong\; R.$$

Take W to be the Golod-Shafarevich ring and take R to be a prime locally finite ring. Then S/Q is an algebraic algebra.

e.g. Take

$$R = \left\{ \begin{pmatrix} C & 0 & \cdots \\ 0 & C & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \right\}.$$

Notice that R is primitive, locally finite and hence S/Q is an affine algebraic algebra that is primitive. It is not finite dimensional, however, as it has the Golod-Shafarevich ring as a homomorphic image.