Recent developments in 0-1 laws for graphs

Limit laws for graphs

Some interesting questions:

- What is the probability that a forest consists of a single tree?

- What is the probability that a (rooted unlabeled) binary forest consists of a single tree?

- What is the probability that a graph is connected?

Given a suitable family of graphs (maps, etc.). We let

$$c(n) = \# \text{ connected objects of size } n$$

and

$$a(n) = \# \text{ objects of size } n.$$

We let $C(x)$ be the generating function of the $c(i)$ and let $A(x)$ be the generating function of the $a(i)$.

If we are in the labeled case, we take expontential generating functions; i.e.,

$$C(x) = \sum_{n \geq 0} c(n)x^n/n!, \qquad A(x) = \sum_{n \geq 0} a(n)x^n/n!.$$

If we are dealing with the unlabeled case, we take the ordinary generating functions; i.e.,

$$C(x) = \sum_{n \geq 0} c(n)x^n, \qquad A(x) = \sum_{n \geq 0} a(n)x^n.$$

Generating function relations

LABELED CASE:

$$A(x) = \mathrm{exp}\,(C(x)).$$

UNLABELED CASE:

$$\begin{aligned} A(x) &= \prod_{n\geq 1} (1 - x^n)^{-c(n)} \\ &= \mathrm{exp}\,(C(x) + C(x^2)/2 + C(x^3)/3 + \cdots). \end{aligned}$$

Probabilities

In general, given a property P, and a suitable family of graphs, $\mathcal{F}$, we say that the probability that a graph in $\mathcal{F}$ has property P is:

$$\lim_{n\to\infty} \frac{\#\text{ graphs of size } n \text{ in } \mathcal{F} \text{ with property } P}{a(n)},$$

whenever the above limit exists.

For example, the probability that a graph is connected is:

$$\lim_{n\to\infty} c(n)/a(n),$$

whenever the above limit exists.

Key Results

Let R denote the radius of convergence of $C(x)$.

- If $R > 0$ and $C(R) = \infty$, then
$$\lim_{n\to\infty} c(n)/a(n) = 0$$
(if the limit exists).

- If $R > 0$ and $C(R) < \infty$, then
$$0 < \lim_{n\to\infty} c(n)/a(n) < 1$$
(if the limit exists).

- If $R = 0$, then
$$\lim_{n\to\infty} c(n)/a(n) = 1$$
(if the limit exists).

EXAMPLES:

Labeled, rooted forests. $c(n) = n^{n-1}$ and

$$C(x) = \sum_{n=1}^{\infty} n^{n-1} x^n / n!.$$

Radius of convergence: $R = 1/e$. $C(1/e) < \infty$.

We have

$$\lim_{n \to \infty} c(n)/a(n) = 1/A(1/e) = \exp(-C(1/e)) = 1/e.$$

Unlabeled rooted binary forests.

$$c(n) \;=\; \binom{2n}{n}/(n+1), \qquad R = 1/4.$$

We have

$$\begin{aligned} \lim_{n\to\infty} c(n)/a(n) \;&=\; 1/A(1/4) \\ &=\; \prod_{n\geq 1} (1 - 1/4^n)^{\binom{2n}{n}/(n+1)} \\ &\cong\; .5767. \end{aligned}$$

Labeled graphs. $a(n) = 2^{\binom{n}{2}}$. Hence $R = 0$. We have

$$\lim_{n\to\infty} c(n)/a(n) \;=\; 1.$$

How do we know the probability of connectedness exists?

Some remarks about first order logic

- Variables, x, y, z, ..., represent vertices.

- We can express equality, $x = y$.

- We can express adjacency $x \sim y$.

- We may use Boolean connectives, $\vee, \wedge, \neg$.

Example We can express the fact that a graph G has a triangle as a subgraph in first order logic:

$$\exists x, y, z \in G,\ x \sim y \wedge y \sim z \wedge z \sim x.$$

We can express the fact that a graph G has no isolated points:

$$\forall x \in G, \exists y \in G \text{ such that } x \sim y.$$

In Monadic Second Order logic, we may quantify over subsets.

Example We can express the fact that a graph G is connected in MSO logic.

$$\begin{array}{ll} & \exists\, U, V \subseteq G, \text{ such that} \\ & (\forall x \in G, (x \in U) \vee (x \in V)) \\ \wedge & (\forall x \in G, \neg(x \in U) \wedge (x \in V)) \\ \wedge & (\forall x \in U, \forall y \in V, \neg x \sim y) \\ \wedge & (U \neq \emptyset) \wedge (V \neq \emptyset). \end{array}$$

THEOREM 1: (Compton) If

$$\lim_{n\to\infty} a(n-1)/a(n) \to 1,$$

then any statement in MSO logic is true with probability 0 or 1.

THEOREM 2: If $c(n) = O(n^k)$ for some k and $\gcd(\{n \mid c(n) \neq 0\}) = 1$, then

$$\lim_{n \to \infty} a(n-1)/a(n) \to 1.$$

COROLLARY If the conditions in THEOREM 2 hold, the probability of connectedness exists (it must be 0).

PARTITIONS

$$c(n) = 1 \qquad a(n) = p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}).$$

THEOREM 3: (Compton) If

$$\lim_{n\to\infty} a(n-1)/a(n)$$

exists and is equal to $R < 1$, and $a(n)R^n$ is eventually nondecreasing, then any statement in MSO logic is true with some limiting probability.

THEOREM 4: If $c(n-1)/c(n) \to R < 1$ and $c(n)R^n n > \lambda > 1$ for all n sufficiently large, then the conditions of THEOREM 3 are satisfied.

THEOREM 5: If $c(n-1)/c(n) \to R$, then $a(n-1)/a(n) \to R$.

Theorem 5 is in fact a conjecture due to Gregory Freiman and Boris Granovsky, who formulated this conjecture during their research into probability and distributions.

A SURPRISING ANALOGUE

ADDITIVE:

- Graphs
- Maps

MULTIPLICATIVE:

- Integers
- Abelian groups

In the multiplicative case, we have some collection in which every element can be uniquely decomposed into “primes”. If

$$X = P_1^{m_1} \times \cdots \times P_k^{m_k},$$

then

$$\text{size}(X) = \prod_{i=1}^{k} \text{size}(P_i)^{m_i}.$$

In the multiplicative case, we again let

$$c(n) \;=\; \# \text{ of ``primes'' of size } n$$

and

$$a(n) \;=\; \# \text{ of objects of size } n.$$

In the multiplicative case, we form Dirichlet series:

$$C(s) := \sum_{n \geq 2} c(n)/n^s.$$

$$A(s) = \sum_{n \geq 1} a(n)/n^s = \prod_{j \geq 2} (1 - j^{-s})^{-c(j)}.$$

POWER SERIES

Circle of convergence. $|z| < R$

DIRICHLET SERIES

Abscissa of convergence. Re$(s) < \alpha$**.**

In the power series case, we looked at expressions such as:

$$\lim_{n\to\infty} c(n)/a(n).$$

In the Dirichlet series case, the analogue is to look at

$$\lim_{n\to\infty} \frac{c(1)+\cdots+c(n)}{a(1)+\cdots+a(n)}.$$

Key Results

Let R denote the radius of convergence of $C(x)$.

- If $R > 0$ and $C(R) = \infty$, then

$$\lim_{n \to \infty} c(n)/a(n) = 0$$

(if the limit exists).

- If $R > 0$ and $C(R) < \infty$, then

$$0 < \lim_{n \to \infty} c(n)/a(n) < 1$$

(if the limit exists).

- If $R = 0$, then

$$\lim_{n \to \infty} c(n)/a(n) = 1$$

(if the limit exists).

Multiplicative Key Results

Let α denote the abscissa of convergence of $C(s)$.

- If $\alpha < \infty$ and $C(\alpha) = \infty$, then

  $$\lim_{n\to\infty} \frac{c(1) + \cdots + c(n)}{a(1) + \cdots + a(n)} \;=\; 0$$

  (if the limit exists).

- If $\alpha < \infty$ and $C(\alpha) < \infty$, then

  $$0 < \lim_{n\to\infty} \frac{c(1) + \cdots + c(n)}{a(1) + \cdots + a(n)} < 1$$

  (if the limit exists).

- If $\alpha = \infty$, then

  $$\lim_{n\to\infty} \frac{c(1) + \cdots + c(n)}{a(1) + \cdots + a(n)} = 1$$

  (if the limit exists).

ANALOGUES OF COMPTON'S THEOREMS

THEOREM: Let $S(x) = \sum_{n \le x} a(n)$. If

$$S(x/2)/S(x) \to 1,$$

then we have a MSO $0 - 1$-law.

THEOREM: Let $S(x) = \sum_{n \le x} a(n)$. If

$$S(x/2)/S(x) \to 2^{-\alpha}$$

and $S(x)x^{-\alpha}$ is eventually nondecreasing, then we have a MSO logical limit law.