A proof of a partition conjecture of Bateman and Erdős

Observation: If

$$\lambda_1 \geq \cdots \geq \lambda_k \geq 1$$

satisfies

$$\lambda_1 + \cdots + \lambda_k = n,$$

then we can obtain a partition of $n + 1$ by simply adding 1; that is,

$$\lambda_1 + \cdots + \lambda_k + 1 \; = \; n + 1$$

is a partition of $n + 1$. We therefore have

$$p(n+1) - p(n) \; \geq \; 0 \qquad \text{for all } n \geq 0.$$

TWO NATURAL QUESTIONS:

1. Are the k^{th} differences of the partitions eventually positive?

2. If so, then what happens if we impose restrictions upon the patitions?

Bateman and Erdős (1956) answered these questions completely.

Their motivation was to improve existing Tauberian theorems.

Given a subset A of $\{1, 2, 3, \cdots\}$, let $p_A(n)$ denote the number of partitions of n with parts from A; i.e.,

$$p_A(n) \;=\; [x^n] \prod_{a \in A} (1 - x^a)^{-1}.$$

Let $p_A^{(k)}(n)$ denote the k^{th} difference of $p_A(n)$; i.e.,

$$p_A^{(k)}(n) \;=\; [x^n](1 - x)^k \prod_{a \in A} (1 - x^a)^{-1}.$$

For example,

$$p_A^{(1)}(n) \;=\; p_A(n) - p_A(n-1),$$

$$\begin{aligned} p_A^{(2)}(n) \;&=\; p_A^{(1)}(n) - p_A^{(1)}(n-1) \\ &=\; p_A(n) - 2p_A(n-1) + p_A(n-2). \end{aligned}$$

DEFINITION: We say a set $A \subseteq \{1, 2, 3, \ldots\}$ has property P_k if:

1. $|A| > k$; and

2. $\gcd(A \setminus \{a_1, \ldots, a_k\}) = 1$ for any $a_1, \ldots, a_k \in A$.

Bateman and Erdős showed the following remarkable fact: $p_A^{(k)}(n)$ is eventually positive **if and only if** A has property P_k.

Moreover, they showed that if A has property P_k,

$$p_A^{(k+1)}(n)/p_A^{(k)}(n) \;\to\; 0 \qquad \text{as } n \to \infty.$$

When $A = \{1, 2, 3, \ldots\}$, Rademacher's formula for the number of partitions of n gives

$$p_A^{(k+1)}(n)/p_A^{(k)}(n) \sim \pi/\sqrt{6n}.$$

It seems therefore reasonable to expect the following conjecture.

Conjecture. (Bateman-Erdős) If A has property P_k,

$$p_A^{(k+1)}(n)/p_A^{(k)}(n) = O(n^{-1/2}).$$

The proof of this conjecture appears in the Journal of Number Theory **87** (2001) 144–153.

The first step in proving this conjecture is the following lemma.

Lemma. Let

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} f(n)x^n \\ G(x) &= \sum_{n=0}^{\infty} g(n)x^n = (1-x)^{-1}F(x), \text{ and} \\ H(x) &= \sum_{n=0}^{\infty} h(n)x^n = (1-x)^{-2}F(x) \end{aligned}$$

be three power series; moreover, suppose these power series have nonnegative coefficients. Then

$$nf(n) = \mathrm{O}(h(n)) \Longrightarrow n^{1/2}g(n) = \mathrm{O}(h(n)).$$

This lemma allows us to work in the ring of formal power series.

WHY?

$$xF'(x) = \sum_{n=0}^{\infty} nf(n)x^n.$$

Thus to prove $g(n) = O(h(n)n^{-1/2})$ it suffices to show that $xF'(x) \leq CH(x) + p(x)$, for some constant C and some polynomial $p(x)$, where the inequality is taken coefficient-wise.

Let A be a subset of the positive integers. Let

$$\begin{aligned} H(x) &= \sum_{n=0}^{\infty} p_A(n)x^n = \prod_{a\in A}(1-x^a)^{-1} \\ G(x) &= \sum_{n=0}^{\infty} p_A^{(1)}(n)x^n, \text{ and} \\ F(x) &= \sum_{n=0}^{\infty} p_A^{(2)}(n)x^n. \end{aligned}$$

GOAL: To show $p_A^{(1)}(n)/p_A(n) = O(n^{-1/2})$ when A has property P_0; i.e., we must show $g(n)n^{1/2} = O(h(n))$ when $\gcd(A) = 1$.

To do this, we use the lemma and compare the coefficients of $xF'(x)$ to the coefficients of $H(x)$.

Recall

$$F(x) = (1-x)^2 \prod_{a \in A} (1-x^a)^{-1}.$$

We have

$$xF'(x) = F(x)(-2x/(1-x) + \sum_{a \in A} ax^a/(1-x^a)).$$

ASIDE

$$\sum_{a\geq 1} ax^a/(1-x^a) \;=\; \sum_{n=1}^{\infty} \sigma(n)x^n.$$

Unfortunately, $\sigma(n)$ is not very well-behaved. Its sequence of partial sums, however, is very well-behaved.

$$\sigma(1)+\sigma(2)+\cdots+\sigma(n) \;\sim\; Cn^2.$$

We have

$$\begin{aligned} &(1-x)^{-1}\Big(\sum_{n=1}^{\infty} \sigma(n)x^n\Big) \\ = \;& \sigma(1)x \;+\; (\sigma(1)+\sigma(2))x^2 \\ &+ (\sigma(1)+\sigma(2)+\sigma(3))x^3+\cdots. \end{aligned}$$

We have

$$(1-x)^{-1}\sum_{a\in A} ax^a/(1-x^a) \leq \sum_{n=1}^{\infty} Cn^2x^n,$$

for some C. Therefore

$$\begin{aligned} & xF'(x) \\ = \; & F(x)(-2x/(1-x)+\sum_{a\in A} ax^a/(1-x^a)) \\ \leq \; & F(x)(1-x)(-2x/(1-x)^2+\sum_{n=1}^{\infty} Cn^2x^n) \\ = \; & F(x)(1-x)(\sum_{n=1}^{\infty}(Cn^2-2n)x^n) \end{aligned}$$

Now

$$[x^n]2C/(1-x)^3 \;=\; C(n+2)(n+1) \;\geq\; (Cn^2-2n)$$

$$\begin{aligned} xF'(x) \;&\leq\; F(x)(1-x)(\sum_{n=1}^{\infty}(Cn^2-2n)x^n \\ &\leq\; F(x)(1-x)(2C/(1-x)^3) \\ &=\; 2CF(x)(1-x)^{-2} \\ &=\; 2CH(x). \end{aligned}$$

Taking the coefficient of x^n we see

$$nf(n) = O(h(n)),$$

or equivalently,

$$\begin{aligned} np_A^{(2)}(n) \;&=\; O(p_A(n)) \\ &\Longrightarrow\; p_A^{(1)}(n) \;=\; O(p_A(n)n^{-1/2}). \end{aligned}$$