Examples in finite Gel′fand-Kirillov dimension II

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Abstract

For any field F, we give an example of a prime finitely generated F-algebra of GK dimension 2 for which GK dimension is not finitely partitive. For each $m \in \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$, we give an example of a prime finitely generated F-algebra having GK dimension 2 which has classical Krull dimension m; the algebra of GK dimension 2 which has infinite classical Krull dimension has the additional property that it does not satisfy the ascending chain condition on prime ideals. This answers some questions of Bergman.

1 Introduction

Given a finitely generated algebra A over a field F, we define the *GK dimension* of A to be

$$\mathrm{GKdim}(A) \ := \ \limsup_{n\to\infty} \log(\dim(V^n))/\log n, \tag{1.1}$$

where V is a finite dimensional F-vector space which contains 1_A and generates A as an F-algebra. (We note that this definition is independent of choice of vector space V meeting the criteria mentioned above.) If T is an algebra which is not finitely generated, we define the GK dimension of T to be the supremum of the GK dimensions of its finitely generated subalgebras. Given an algebra A and a finitely generated right A-module M, we define the GK dimension of M, denoted by $\mathrm{GKdim}(M_A)$ or $\mathrm{GKdim}(M)$, to be

$$\mathrm{GKdim}(M) \ := \ \sup_{V,X} \ \limsup_{n\to\infty} \log\left(\dim(XV^n)\right)/\log n,$$

where V ranges over all finite dimensional subspaces of A which contain 1_A and X ranges over all finite dimensional subspaces of M. Basic properties of GK dimension can be found in [3].

In [1], a procedure is given which, given a countably generated prime algebra T, allows one to construct a finitely generated prime algebra $\mathcal{A}(T, F; \Phi)$ such that T is a corner of $\mathcal{A}(T, F; \Phi)$ and the GK dimension of A is exactly two greater than the GK dimension of T. We now briefly describe this construction, due to Small (unpublished). Let F be a field and let T be a prime, countably generated F-algebra. We construct the *affinization* of T as follows. Let $R = F\{x, y\}$ and let

$$S \;=\; \begin{pmatrix} F + Ry & R \\ Ry & R \end{pmatrix}.$$

The ring S is generated as an F-algebra by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \text{ where } a \in \{1, x, y\}. \tag{1.2}$$

Hence S is an affine F-algebra. $F + Ry$ is a free F-algebra on the infinitely many generators $\{x^i y \mid i \geq 0\}$. It follows that we have a surjective ring homomorphism

$$\Phi : F + Ry \;\rightarrow\; T. \tag{1.3}$$

Let

$$P \;=\; \ker(\Phi) \tag{1.4}$$

and let $e_{i,j}$ denote the matrix with a 1 in the (i, j) entry and zeros everywhere else. Notice P is a prime ideal. Observe that $Q' := S(Pe_{1,1})S$ satisfies $e_{1,1}Q'e_{1,1} = P$. Using Zorn's lemma we can choose an ideal Q in S maximal with respect to the property that

$$e_{1,1}Qe_{1,1} \;=\; \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.5}$$

By maximality, we have that Q is prime; moreover Q is uniquely determined due to maximality.

$$Q \;\supseteq\; \begin{pmatrix} P & PR \\ RyP & RyPR \end{pmatrix}. \tag{1.6}$$

The algebra S/Q has the property that

$$\overline{e_{1,1}}(S/Q)\overline{e_{1,1}} \;\cong\; T. \tag{1.7}$$

We call S/Q the *affinization of T with respect to* Φ and denote it by

$$\mathcal{A}(T, F; \Phi). \tag{1.8}$$

We note that since the prime ideal Q is uniquely determined by Φ, the algebra $\mathcal{A}(T, F; \Phi)$ is uniquely determined by T, F, and Φ. Let

$$\mathcal{M} \ = \ \{m_1, m_2, \ldots\} \subseteq \ \mathbb{N} \tag{1.9}$$

be a set with the property that

$$m_{i+1} \ \geq \ 3m_i \qquad \text{for } i \geq 1. \tag{1.10}$$

Throughout, we shall assume that $\Phi : F + F\{x, y\}y \to T$ satisfies:

- For each $i \notin \mathcal{M}$, we have

$$\Phi(x^i y) = 0 \text{ if } i \notin \mathcal{M}; \tag{1.11}$$

- for any m, the set

$$\{\Phi(x^i y) \mid i \geq m\} \tag{1.12}$$

  spans T as an F-vector space.

Under these conditions, we have the following theorem.

Theorem 1.1 *We have*

- $e_{2,2}Qe_{2,2} = RyPRe_{2,2}$.
- $\mathrm{GKdim}(\mathcal{A}(T, F; \Phi)) = \mathrm{GKdim}(T) + 2$.

Proof. See [1]. ∎

This result was used to construct some examples of rings with counter-intuitive properties. There are, however, additional examples which can be constructed with these methods. We use this result to show the following.

Theorem 1.2 *We have the following examples:*

- *there is a finitely generated prime algebra for which GK dimension is not finitely partitive; and*
- *for $m \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ there is a finitely generated prime algebra of GK dimension 2 and classical Krull dimension m; moreover, the algebra with infinite classical Krull dimension does not satisfy acc on prime ideals.*

2 Examples

Example 2.1 *An algebra for which GK dimension is not finitely partitive.*

In 8.3.17 of [4] it is asked whether there exist affine algebras for which GK dimension is not finitely partitive. Before proceeding, we must explain what it means for an algebra to be finitely partitive.

Definition 2.1 *Let A be an affine algebra with finite GK dimension and let M be a finitely generated A module of integer GK dimension. We say that GK dimension is* finitely partitive *for M if there exists some n such that for any descending chain of finitely generated A-submodules of M,*

$$M_0 = M \supseteq M_1 \supseteq M_2 \cdots \supseteq M_m,$$

with the property that $\mathrm{GKdim}(M_{i-1}/M_i) = \mathrm{GKdim}(M)$ *for* $1 \leq i \leq m$*, we have $m < n$. We say that GK dimension is* finitely partitive *for A if it is finitely partitive for every finitely generated module M of integer GK dimension.*

Intuitively, we think of the relation $\mathrm{GKdim}(M/N) = \mathrm{GKdim}(M)$ as expressing the fact that N is a relatively "small" submodule of M. Thus finite partitivity roughly says that if we start with a finitely generated A-module and take a small submodule of this module and then take a small submodule of the new submodule and continue this process, then this process must terminate at some point.

We construct an algebra A of GK dimension 2 for which GK dimension is not finitely partitive; moreover, we show that the modules M_i can be chosen to be cyclic A-modules. We need a simple remark about the GK dimension of modules.

Remark 2.2 *Let A be a finitely generated F-algebra and let M be a finitely generated right A-module.*

- *If $0 \to N \to M \to L \to 0$ is a short exact sequence of right A-modules, then* $\mathrm{GKdim}(M) \geq \sup(\mathrm{GKdim}(N), \mathrm{GKdim}(L))$.
- *If $M_1, \ldots, M_n$ are right A-modules, then* $\mathrm{GKdim}(M_1 + \cdots + M_n) = \sup_{1 \leq i \leq n} \mathrm{GKdim}(M_i)$.

- *If M is a right A-module and $\alpha : M \to M$ is an injective homomorphism, then* $\mathrm{GKdim}(M/\alpha(M)) \le \mathrm{GKdim}(M) - 1$.

Proof. See Proposition 5.1 on page 52 of [3]. ∎

Lemma 2.3 *Let A be an F-algebra with a finite set of elements $\{e_{i,j}\}_{i,j=1}^n$ which satisfy $e_{i,j}e_{k,\ell} = \delta_{j,k}e_{i,\ell}$ for $1 \le i,j,k,\ell \le n$. Then* $\mathrm{GKdim}(e_{i,i}A) = \mathrm{GKdim}(eA)$ *for $1 \le i \le n$, where $e = e_{1,1} + \cdots + e_{n,n}$.*

Proof. Observe that $e_{i,j}A \hookrightarrow e_{i,i}A$ and hence

$$\mathrm{GKdim}(e_{i,j}A) \ \le \ \mathrm{GKdim}(e_{i,i}A) \tag{2.13}$$

by Remark 2.2. We have a surjective homomorphism of right A-modules

$$e_{i,j}A \ \to \ e_{j,j}A$$

given by $e_{i,j}a \mapsto e_{j,i}e_{i,j}a = e_{j,j}a$. Hence

$$\mathrm{GKdim}(e_{j,j}A) \ \le \ \mathrm{GKdim}(e_{i,j}A) \ \le \ \mathrm{GKdim}(e_{i,i}A)$$

by Remark 2.2 and equation (2.13). By symmetry, we have

$$\mathrm{GKdim}(e_{1,1}A) \ = \ \mathrm{GKdim}(e_{2,2}A) \ = \ \cdots \ = \ \mathrm{GKdim}(e_{n,n}A).$$

But

$$eA \cong e_{1,1}A \oplus e_{2,2}A \oplus \cdots \oplus e_{n,n}A$$

as right A-modules and hence

$$\mathrm{GKdim}(eA) \ = \ \sup_{1\le i\le n} \mathrm{GKdim}(e_{i,i}A)$$

by Remark 2.2. The result follows. ∎

We now begin our construction. Let F be a field and let T_n be the subring of row finite $\aleph_0 \times \aleph_0$ matrices consisting of all matrices of the form

$$\begin{pmatrix} X & 0 & \cdots & 0 \\ 0 & X & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where X is an $n \times n$ matrix. We take

$$T \ = \ \bigcup_{j=1}^{\infty} T_j. \tag{2.14}$$

Lemma 2.4 *The algebra T is a countably infinite dimensional, simple, and has GK dimension* 0.

Proof. We have

$$T \;=\; \bigcup_{j=1}^{\infty} T_j.$$

Since $T_n \cong M_n(F)$, we see that T is countably infinite dimensional over F. Since any finite dimensional vector subspace of T must be in T_n for some n, we conclude that T has GK dimension 0. Finally, let $a \in T$. Then $a \in T_n$ for some n and hence $a \in T_m$ for all multiples m of n. Since T_m is simple, TaT must contain T_m for all multiples m of n. Hence

$$TaT \;\supseteq\; \bigcup_{j=1}^{\infty} T_{jn} \;=\; T.$$

Thus T is simple. ∎

Since T is a countably generated ring of GK dimension 0, there exists an affinization $\mathcal{A}(T, F; \Phi)$ of GK dimension 2. We identify T with the ring $\overline{e_{1,1}}\mathcal{A}(T, F; \Phi)\overline{e_{1,1}}$.

Lemma 2.5 *The $\mathcal{A}(T, F; \Phi)$-module $\overline{e_{1,1}}\mathcal{A}(T, F; \Phi)$ has GK dimension* 1 *as a right $\mathcal{A}(T, F; \Phi)$-module.*

Proof. Let $A = \mathcal{A}(T, F; \Phi)$. Observe that we have a map

$$\overline{e_{1,1}}A \;\rightarrow\; \overline{ye_{2,1}}A$$

given by $\overline{e_{1,1}}a \mapsto \overline{ye_{2,1}e_{1,1}}a$. Choose $\alpha \in F + F\{x, y\}y$ and $\beta \in F\{x, y\}$ such that $\overline{e_{1,1}}a = \overline{\alpha e_{1,1}} + \overline{\beta e_{1,2}}$. To see that this map is injective, suppose that $\overline{ye_{2,1}}a = 0$. Multiplying on the left by $\overline{x^i e_{1,1}}$ and multiplying on the right by $\overline{x^j y e_{2,1}}$ gives that $x^i y \alpha x^j y \in P$ for all i and j. Thus $\alpha \in P$ by item (1.12) and so $\overline{e_{1,1}}a = \overline{\beta e_{1,2}}$. Now since $\overline{ye_{2,1}}a = 0$, we have $\overline{y\beta e_{2,2}} = 0$ and so $y\beta \in RyPR$ by Theorem 1.1. Thus $\beta \in PR$ and so $\overline{e_{1,1}}a = 0$ by equation (1.6). By Theorem 1.1, we may assume that

$$B \;:=\; \overline{e_{2,2}}\mathcal{A}(T, F; \Phi)\overline{e_{2,2}} \cong R/RyPR$$

and has GK dimension 2. Now $\overline{e_{2,2}}A$ is a left $A^{\rm op}$-module and a right B-module. It is generated as a B-module by $\overline{ye_{2,1}}$ and $\overline{ye_{2,2}}$. Thus by Corollary 5.4 on page 55 of [3], we have

$$\mathrm{GKdim}(\overline{ye_{2,2}}A_A) \;=\; \mathrm{GKdim}(\overline{ye_{2,2}}A_B).$$

Now $\overline{ye_{2,2}}A$ is a homomorphic image of $\overline{y}B \oplus \overline{y}B$ as right B-modules and hence

$$\mathrm{GKdim}(\overline{ye_{2,2}}A_B) \;\le\; \mathrm{GKdim}(\overline{y}B_B).$$

We identify B with $R/RyPR$. Observe that we have an injection of right B-modules $\overline{y}B \hookrightarrow B/\overline{x}B$, given by $\overline{y}b \mapsto \overline{y}b + \overline{x}B$. The fact that this map is an injection follows from the fact that

$$RyPR \;=\; \bigoplus_{i=0}^{\infty} x^i yPR. \tag{2.15}$$

Thus $\mathrm{GKdim}(\overline{y}B) \le \mathrm{GKdim}(B/\overline{x}B)$. Finally, using equation (2.15), observe that the image of x is right-regular in B and hence

$$\mathrm{GKdim}(B/\overline{x}B) \;\le\; \mathrm{GKdim}(B) - 1 \;\le\; 1$$

by Remark 2.2. On the other hand, since $\overline{e_{1,1}}A$ is an infinite, principal right ideal, and A is finitely generated as an F-algebra, we see that $\overline{e_{1,1}}A$ has GK dimension at least 1 as a right A-module. The result follows. ∎

Let $E_{i,j}(n)$ denote the matrix in T whose $(k,\ell)^{\rm th}$ entry is 1 if $k - i = \ell - j = nm$ for some nonnegative integer m and is zero otherwise. Given integers j and n with $1 \le j \le n$, define $I_{j,n}$ to be the right ideal of $\mathcal{A}(T, F, \Phi)$ generated by

$$\begin{pmatrix} E_{j,j}(n) & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $e_{1,1}\mathcal{A}(T, F, \Phi) \cong I_{1,1}$ as right $\mathcal{A}(T, F; \Phi)$-modules.

Proposition 2.6 *Let n be a positive integer and let j be a positive integer less than or equal to n. Then $I_{j,n}$ has GK dimension 1 as an $\mathcal{A}(T, F; \Phi)$-module.*

Proof. By Lemma 2.5, $I_{1,1} \cong \overline{e_{1,1}}\mathcal{A}(T,F;\Phi)$ has GK dimension 1. Observe that $\{E_{i,j}(n)\overline{e_{1,1}}\}_{1\le i,j\le n}$ satisfies the conditions of Lemma 2.3 and hence $I_{j,n} = E_{j,j}(n)\overline{e_{1,1}}\mathcal{A}(T,F;\Phi)$ has the same GK dimension as

$$(E_{1,1}(n) + \cdots + E_{n,n}(n))\overline{e_{1,1}}\mathcal{A}(T,F;\Phi) \;=\; \overline{e_{1,1}}\mathcal{A}(T,F;\Phi) \;=\; I_{1,1}.$$

The result follows. ∎

Corollary 2.7 *GK dimension is not finitely partitive for $\mathcal{A}(T,F;\Phi)$.*

Proof. Observe that

$$I_{j,2n} \oplus I_{j+n,2n} \;\cong\; I_{j,n} \tag{2.16}$$

via the isomorphism $(a_1, a_2) \mapsto a_1 + a_2$. Consider the descending chain of $\mathcal{A}(T,F;\Phi)$-modules,

$$I_{1,1} \;\supseteq\; I_{2,2} \;\supseteq\; I_{4,4} \;\supseteq\; \cdots.$$

By equation (2.16)

$$I_{2^n,2^n}/I_{2^{n+1},2^{n+1}} \cong I_{2^n,2^{n+1}}$$

and hence

$$\mathrm{GKdim}\Big((I_{2^n,2^n}/I_{2^{n+1},2^{n+1}})_{\mathcal{A}(T,F;\Phi)}\Big) \;=\; 1 \;=\; \mathrm{GKdim}((I_{1,1})_{\mathcal{A}(T,F;\Phi)}).$$

Since $I_{1,1}$ is a finitely generated $\mathcal{A}(T,F;\Phi)$ of GK dimension 1, we see that $\mathcal{A}(T,F;\Phi)$ is not finitely partitive. ∎

Example 2.2 *Affine algebras of GK dimension 2 with classical Krull dimension m for $m \in \mathbb{N}$.*

We begin by giving a procedure which, given a ring, allows one to create a new ring with larger classical Krull dimension. Let R be a ring. We define $T_n(R)$ to be be the set of all row-finite $\aleph_0 \times \aleph_0$ matrices consisting of an $n\times n$ strictly upper-triangular matrix repeated down the diagonal. We define $I = I(R)$ to be the union of the $T_n(R)$. We identify R with the subring of all row-finite scalar $\aleph_0 \times \aleph_0$ matrices with entries in R. Under this identification, we define

$$G(R) \;:=\; R + I(R). \tag{2.17}$$

Lemma 2.8 *Let R be a prime ring of finite classical Krull dimension. Then $G(R)$ is a prime ring of classical Krull dimension one greater than the classical Krull dimension of R; moreover* $\mathrm{GKdim}(G(R)) = \mathrm{GKdim}(R)$.

Proof. We identify R with the subring of $G(R)$ consisting of scalar matrices of $G(R)$ and write $G(R) = R + I$, where I is the ideal of $G(R)$ consisting of the strictly upper-triangular matrices of $G(R)$. Given $i \leq j \leq N$, let $E_{i,j}(N)$ denote the element of $G(R)$ which has a 1 in the $(i+Nk, j+Nk)^{\mathrm{th}}$ entry for all $k \geq 0$ and a zero in every other entry.

We first show that $G(R)$ is a prime ring. Let $u, v \in G(R)$ be nonzero. Pick (i,j) and (k,ℓ) such that the $(i,j)^{\mathrm{th}}$ entry of u is nonzero and the $(k,\ell)^{\mathrm{th}}$ entry of v is nonzero. Pick n such that $u, v \in T_n(R)$. There is some positive integer p such that $k + pn > j$. Then $uE_{j,k+pn}(n(p+1))v \in G(R)$ has a nonzero $(i, \ell + pn)^{\mathrm{th}}$ entry. Thus we see that $G(R)$ is prime.

We now consider the classical Krull dimension of $G(R)$. Let P be a nonzero prime ideal of $G(R)$. Assume first that $P \cap R$ is a zero ideal of R. Then $P \subseteq I$. Hence there is some n and some $y \in T_n(R)$ such that

$$y = \begin{pmatrix} C & 0 & 0 & \cdots \\ 0 & C & 0 & \cdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in P,$$

With $C = (c_{i,j})_{i,j=1}^{n}$, $c_{i,j} \in R$. There is some index (i_0, j_0) such that $c_{i_0,j_0} \in R$ is nonzero. Let (i,j) be such that $j - i \geq 2n+1$. Choose p such that $i_0 + pn > i \geq i_0 + n(p-1)$. Since $j \leq i - (2n+1)$, we have

$$j \geq i + (2n+1) \;\geq\; i_0 + n(p-1) + 2n + 1 \;>\; (i_0 + n) + np \;>\; j_0 + np.$$

Choose m such that $nm > j$. Then

$$E_{i,i_0+pn}(nm)yE_{j_0+pn,j}(nm) \;=\; c_{i_0,j_0}E_{i,j}(nm).$$

Now

$$c_{i,j}G(R)E_{i,j}(nm) \subseteq P.$$

Since $c_{i,j} \in R \subseteq G(R)$ is not in P, we see that $E_{i,j}(nm) \in\subseteq P$ whenever $j - i \geq 2n+1$. Thus $I^{2n+1} \subseteq P$. Since P is prime, we see that $I \subseteq P$. Thus $G(R)/P$ is a homomorphic image of $G(R)/I \cong R$. Thus the length of any chain beginning $0 \subsetneq P$ is at most the one more that the classical Krull dimension of R. We now complete the proof by induction on the classical Krull dimension of R. Suppose that the classical Krull dimension of R is zero. Let P be a nonzero prime ideal of $G(R)$. Then $P \cap R \subseteq G(R)$ must be zero and so we have that the classical Krull dimension of $G(R)$ is 1. Now suppose that the claim is true for all rings of classical Krull dimension less than n and suppose that R has classical Krull dimension n. Let P be a nonzero prime ideal of $G(R)$. Then we may assume that $P \cap R \subseteq G(R)$ is a nonzero ideal of R. Then $G(R)/P$ is a homomorphic image of $G(R/R \cap P)$. Since $R/R \cap P$ is a prime homomorphic image of R of strictly smaller classical Krull dimension than R, we see by the inductive hypothesis that any chain of prime ideals of $G(R)$ which begins $0 \subsetneq P$ must have length at most the classical Krull dimension of R. Thus we see that the classical Krull dimension of $G(R)$ is exactly one more than the classical Krull dimension of R.

Finally, to see that $\mathrm{GKdim}(G(R)) = \mathrm{GKdim}(R)$, notice that R is a subring of $G(R)$ and so we have that

$$\mathrm{GKdim}(R) \ \leq \ \mathrm{GKdim}(G(R)).$$

Next let $\{a_1, \ldots, a_m\}$ be a finite subset of $G(R)$. Then there is some n such that $\{a_1, \ldots, a_m\} \subseteq T_n(R)$. Since $T_n(R)$ is isomorphic to a subring of $M_n(R)$, we have that the algebra generated by $a_1, \ldots, a_m$ is a subring of $M_n(R)$ and hence has GK dimension at most $\mathrm{GKdim}(R)$. Thus

$$\mathrm{GKdim}(G(R)) \ \leq \ \mathrm{GKdim}(R).$$

The result follows. ∎

An easy consequence of this lemma is the following.

Corollary 2.9 *Let $R_0 = F$ and let $R_n = G(R_{n-1})$ for $n \geq 1$. Then R_n is a ring of GK dimension* 0 *and Classical Krull dimension n.*

We can also create a prime ring of GK dimension 0 and infinite Krull dimension; in fact, we can even construct an example such that the set of prime ideals do not satisfy the ascending chain condition.

We now construct prime affine algebras of GK dimension 2 having classical Krull dimension $m \in \{2, 3 \ldots\} \cup \{\infty\}$. There are many examples of finitely generated algebras for which the classical Krull dimension is less than the GK dimension. It was, however, unknown whether the classical Krull dimension could be greater than the GK dimension of an affine algebra. In fact, on page 2560 of [2], Bergman asks whether any of the following hold for finitely generated algebras over a field:

- Is the classical Krull dimension less than or equal to the GK dimension?
- If the GK dimension is finite, is the classical Krull dimension finite?
- If the GK dimension is finite, must the collection of prime ideals satisfy the ascending chain condition?

We show that the answer to each of these questions is an emphatic 'no'.

A Goldie ring has classical Krull dimension equal to at most its GK dimension. Also by the Small-Warfield theorem [5], we have that affine prime algebras of GK dimension less than or equal to 1 are PI and hence are Goldie. In particular, they have classical Krull dimension at most 1. For affine algebras of GK dimension 2, the classical Krull dimension can be arbitrarily large (or even infinite). We have the following theorem.

Theorem 2.10 *Let A be an algebra with idempotent e. Let $T = eAe$ Then*

$$\mathrm{clKdim}(T) \ \leq \mathrm{clKdim}(A) \ \leq \ \mathrm{clKdim}(T) + \mathrm{clKdim}(A/(e)) + 1;$$

moreover, if the set of prime ideals of A satisfies the ascending chain condition, then the set of prime ideals of T does too.

Proof. Suppose $Q_1 \subsetneq Q_2$ are two prime ideals in A, with $Q_1 \not\supset (e)$. Observe that $eQ_1e \subseteq eQ_2e$ are two prime ideals in T. Let $P_i = eQ_ie$ for $i \in \{1, 2\}$. We claim that $P_1 \neq P_2$. By passing to the quotient $B = A/Q_1$, we may assume that $Q_1 = P_1 = (0)$. Suppose $P_2 = (0)$. Choose nonzero $a \in Q_2$. Then $eBaBe = (0)$ and hence $e = 0$; i.e., $e \in Q_1$, a contradiction. Let $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$ be an ascending chain of prime ideals of A. To establish the right-most inequality, we may assume that the classical Krull dimension of $A/(e)$ is finite. Let d denote the classical Krull dimension of $A/(e)$. Notice Q_{n-d-1} cannot contain (e), since $A/(e)$ has Classical Krull dimension d. Thus $P_0 \subsetneq P_1 \cdots \subsetneq P_{n-d-1}$ is an ascending chain of prime ideals of T. It follows that the classical Krull dimension of A is at most $d+1$ more than the classical Krull dimension of T. Now suppose that $P_1 \subsetneq P_2$ are two prime ideals in eAe. Observe that $I = AP_1A$ satisfies $eIe = eAP_1Ae = eAeP_1eAe = P_1$. By Zorn's lemma we can find a prime ideal Q_1 maximal with respect to the property that $eQ_1e = P_1$. Similarly, we can find a prime ideal Q_2 maximal with respect to the property that $eQ_2e = P_2$. Now $e(Q_2+Q_1)e = P_2+P_1 = P_2$. Thus by maximality of Q_2 we have that $Q_2 \supseteq Q_1$; moreover, Q_2 strictly contains Q_1 since P_2 strictly contains P_1. It follows that for any chain of prime ideals in T,

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n,$$

we can find a chain of prime ideals in A,

$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n.$$

Hence the classical Krull dimension of A is at least the classical Krull dimension of T, and if the set of prime ideals of T does not satisfy the ascending chain condition, then neither does the set of prime ideals of A. ∎

Corollary 2.11 *Let T be a prime countably generated algebra. Then*

$$\mathrm{clKdim}(T) \le \mathrm{clKdim}(\mathcal{A}(T,F;\Phi) \le \mathrm{clKdim}(T)+2;$$

moreover, if the set of prime ideals of T does not satisfy the ascending chain condition, then neither does the set of prime ideals of $\mathcal{A}(T,F;\Phi)$.

Proof. Let $A = \mathcal{A}(T,F;\Phi)$ and let $e = \overline{e_{1,1}}$. Then $A/(e) \cong F[x]$ which has classical Krull dimension 1. The result follows. ∎

For $m \in \{2,3,4,\ldots\}$, let $A_m = \mathcal{A}(R_m,F;\Phi)$ be an affinization of R_m of GK dimension 2. We then have the following result.

Corollary 2.12 *For $m \in \{2,3,\ldots\}$, there is a homomorphic image of the algebra A_m which is an affine algebra of GK dimension 2 and classical Krull dimension m.*

Proof. We have that A_m has finite Krull dimension $\ge m$. Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ be a maximal chain of prime ideals. We claim that A_m/P_{n-m} has GK dimension 2 and classical Krull dimension m. The fact that A_m/P_{n-m} has classical Krull dimension m is clear. To see that this algebra has GK dimension 2, suppose that this is not the case. Then by Bergman's gap theorem, it must have GK dimension at most 1. Since A_m/P_{n-m} is prime it must be Goldie and hence it has classical Krull dimension at most 1, a contradiction. ∎

Thus we have constructed algebras of GK dimension 2 and classical Krull dimension m for all $m \in \{2, 3, 4, \ldots\}$. We now construct a prime affine F-algebra of GK dimension 2 which has infinite classical Krull dimension; in fact, this algebra does not satisfy acc on prime ideals.

Given an F-algebra B, let $U_n(B)$ denote the subset of $\aleph_0 \times \aleph_0$ row-finite matrices over B of the form

$$\begin{pmatrix} X & 0 & 0 & \cdots \\ 0 & X & 0 & \cdots \\ 0 & 0 & X & \vdots \\ \vdots & \vdots & 0 & \ddots \end{pmatrix},$$

where X is an $n \times n$ upper-triangular matrix of the form $\alpha \mathbf{I}_n + N$, for some $\alpha \in F$ and some nilpotent matrix with entries in B. Define

$$U(B) \; := \; \bigcup_{n \geq 1} U_n(B). \tag{2.18}$$

Lemma 2.13 *Let B be a prime countably generated F-algebra. Then $U(B)$ is a prime countably generated algebra of GK dimension* 0.

Proof. The fact that $U(B)$ is a prime countably generated F-algebra follows using a similar argument as was employed in Lemma 2.8. To see that $U(B)$ has GK dimension 0, let V be a finite dimensional subspace of $U(B)$. By adding F to V if necessary, we can write V as $F + W$, where W is a finite dimensional subspace consisting of strictly upper-triangular matrices. By construction, there exists some m such that $W \subseteq U_m(B)$. Thus $W^m = 0$. Hence $V^n = F + W + W^2 + \cdots + W^m$ for all $n \geq m$, and so we see that $U(B)$ has GK dimension 0. ∎

Theorem 2.14 *There exists an affine F-algebra of GK dimension* 2 *which does not satisfy the ascending chain condition on prime ideals.*

Proof. We take $B = F[x_1, x_2, \ldots]$, a polynomial ring in infinitely many variables. The surjection

$$B \to F[x_\ell, x_{\ell+1}, \ldots]$$

induces a surjective map from $U(B)$ onto $U(F[x_\ell, x_{\ell+1}, \ldots])$. By Lemma 2.13, the kernel of this map is prime, call it P_ℓ; moreover, for $m < n$, P_n is the kernel of the map induced by the composition of surjections

$$U(B) \; \to \; U(F[x_m, x_{m+1}, \ldots]) \to \; U(F[x_n, x_{n+1}, \ldots]).$$

We therefore have

$$P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq \cdots.$$

By Corollary 2.11, any affinization of $U(B)$ does not satisfy acc on prime ideals. Consequently, there exists a map $\Phi : F + F\{x, y\}y \to U(B)$ such that

$$\mathcal{A}(U(B), F; \Phi)$$

is a prime affine F-algebra of GK dimension 2 which does not satisfy acc on prime ideals. ∎

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