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LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER Linear Algebra and its Applications 402 (2005) 101–110
www.elsevier.com/locate/laa

A gap result for the norms of semigroups of matrices

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Received 18 October 2004; accepted 1 December 2004
Available online 28 January 2005
Submitted by R.A. Brualdi

Abstract

Let $\|\cdot\|$ be a matrix norm on $M_d(\mathbb{C})$ and let $\mathscr{A}$ be a finite set of matrices in $M_d(\mathbb{C})$. We define $m_n(\mathscr{A})$ to be the maximum norm of a product of n elements of $\mathscr{A}$. We show that there is a gap in the possible growth of $m_n(\mathscr{A})$, showing that $m_n(\mathscr{A})$ grows either at least exponentially or is bounded by a polynomial in n of degree at most $d-1$. Moreover, we show that the growth is bounded by a polynomial if and only if every element of the semigroup generated by $\mathscr{A}$ has all of its eigenvalues on or inside the unit circle.

AMS classification: Primary 15A60; Secondary 15A18, 15A42, 15A99

Keywords: Semigroups of matrices; Matrix norms; Spectral radius; Growth

1. Introduction

We look at finitely generated semigroups of complex matrices. A *matrix norm*, $\|\cdot\|$, on $M_d(\mathbb{C})$ is just a norm on the vector space $M_d(\mathbb{C})$ which satisfies $\|AB\| \leqslant \|A\| \cdot \|B\|$. A large amount of useful information about matrices and matrix norms can be found in the book of Belitskii and Lyubich [1] and in the book of Hartfiel [5].

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doi:10.1016/j.laa.2004.12.007

We study how the norms of elements of our semigroup grow in terms of their length as a product of some generating set. We make this statement more precise with the next definition.

Definition 1.1. Let $\mathscr{A} = \{A_1, \ldots, A_e\}$ be a finite set of complex matrices and let $\|\cdot\|$ be a matrix norm. We define

$$m_n(\mathscr{A}) := \max_{1\leqslant i_1,\ldots,i_n\leqslant e} \|A_{i_1}\cdots A_{i_n}\|.$$

The quantity $m_n(\mathscr{A})$ has been studied by Daubechies and Lagarias [3,4], who looked at criteria for infinite products of matrices to converge. Given a finite set $\mathscr{A}$, the quantity

$$\limsup_{n\to\infty} |m_n(\mathscr{A})|^{1/n}$$

is called the *joint spectral radius* of $\mathscr{A}$.

Our main theorem is the following.

Theorem 1.2. *Let $\mathscr{A}$ be a finite set of $d \times d$ complex matrices. Then either there is some constant $c > 1$ such that $m_n(\mathscr{A}) > c^n$ for all n sufficiently large, or $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$. Moreover, $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$ if and only if the eigenvalues of every matrix in the semigroup generated by $\mathscr{A}$ are all on or inside the unit circle.*

This theorem says that there is a large gap in the possible growth of $m_n(\mathscr{A})$ for a finite set $\mathscr{A}$ of $d \times d$ complex matrices. The growth of $m_n(\mathscr{A})$ is either at least exponential or it is bounded by a polynomial of degree $d - 1$; moreover it is bounded by a polynomial if and only if the joint spectral radius of $\mathscr{A}$ is at most 1.

So, for example, it is impossible to find a set of matrices $\mathscr{A}$ with $m_n(\mathscr{A}) \sim \exp(\sqrt{n})$.

The main idea we employ is Jacobson's idea of approaching ring theoretic problems via reduction to the simple case. Thus, given a set $\mathscr{A}$ of $d \times d$ complex matrices, we look at the subalgebra of $M_d(\mathbb{C})$ generated by $\mathscr{A}$. We prove our result first in the case that this subalgebra is simple. Then we use this to handle the semisimple case. Finally, we look at the general case using the results we have already accumulated.

2. Proofs

We begin with some notation.

Notation 2.1. Given a $d \times d$ matrix B, we let $B(i, j)$ denote its (i, j)-entry.

Notation 2.2. Given a $d \times 1$ complex vector $\mathbf{v} = (v_1, \ldots, v_d)^{\mathrm{T}}$, we write $|\mathbf{v}|$ for its length; that is, $|\mathbf{v}| = \sqrt{|v_1|^2 + \cdots + |v_d|^2}$.

We now prove a lemma which allows us to characterize sets of matrices $\mathscr{A}$ for which $m_n(\mathscr{A})$ grows exponentially.

Lemma 2.3. *Let $\mathscr{A} = \{A_1, \dots, A_e\}$ be a collection of $d \times d$ complex matrices. Let $\mathscr{S}$ denote the semigroup generated by $\mathscr{A}$. Suppose there exists $B \in \mathscr{S}$ with an eigenvalue λ with $|\lambda| > 1$. Then there is some $c > 1$ such that $m_n(\mathscr{A}) > c^n$ for all n sufficiently large.*

Proof. Write $B = A_{i_m} \cdots A_{i_1}$. Let $\mathbf{v}$ denote an eigenvector of B of unit length corresponding to the eigenvalue λ. Let $\|\cdot\|_{\sup}$ denote the sup norm for $d \times d$ matrices; that is,

$$\|X\|_{\sup} := \sup_{|\mathbf{w}|=1} |X\mathbf{w}|.$$

Since all matrix norms are equivalent, there exist $c_1, c_2 > 0$ such that

$$c_1\|X\|_{\sup} \leqslant \|X\| \leqslant c_2\|X\|_{\sup}$$

for all $d \times d$ matrices X. Let $n > m$. Then we write $n = am + m'$ with $1 \leqslant m' \leqslant m$. Let $X_n = A_{i_{m'}} \cdots A_{i_1} B^a$. Then X_n is a word of length n in $A_1, \dots, A_e$. Define vectors $\mathbf{w}_1, \dots, \mathbf{w}_m$ by

$$\mathbf{w}_j = A_{i_j} \cdots A_{i_1}\mathbf{v}.$$

Notice that $\mathbf{w}_1, \dots, \mathbf{w}_m$ are nonzero since $\mathbf{v}$ is an eigenvector of B corresponding to $\lambda \neq 0$. Thus

$$X_n\mathbf{v} = A_{i_{m'}} \cdots A_{i_1} B^a\mathbf{v} = \lambda^a \mathbf{w}_{m'}.$$

Hence

$$\|X_n\|_{\sup} \geqslant |\lambda|^a |\mathbf{w}_{m'}|.$$

Notice that $a = (n - m')/m \geqslant n/m - 1$. Moreover, there is some $c' > 0$ with $|\mathbf{w}_j| > c'$ for $1 \leqslant j \leqslant m$. Hence

$$\|X_n\|_{\sup} \geqslant (|\lambda|^{1/m})^n c'|\lambda|^{-1}.$$

Hence

$$\|X_n\| \geqslant c_1(|\lambda|^{1/m})^n c'|\lambda|^{-1}.$$

Picking $1 < c < |\lambda|^{1/m}$, we see that $\|X_n\| \geqslant c^n$ for all n sufficiently large. Thus, $m_n(\mathscr{A}) > c^n$ for all n sufficiently large. □

We now study finitely generated semigroups of $d \times d$ complex matrices in which each matrix in the semigroup has all of its eigenvalues on or inside the unit disc. Our ultimate goal being to show that for such semigroups with finite generating set $\mathscr{A}$, $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$. To do this we look at the subalgebra, R, generated by $\mathscr{A}$. We now consider the case that R is the full matrix ring.

Proposition 2.4. *Let $\mathscr{A} = \{A_1, \dots, A_e\}$ be a collection of $d \times d$ complex matrices which generate $M_d(\mathbb{C})$ as a $\mathbb{C}$-algebra; moreover suppose that the semigroup* *generated by $\mathscr{A}$ has the property that every element of this semigroup has all of its eigenvalues on or inside the unit disc. Then $m_n(\mathscr{A})$ is uniformly bounded.*

Proof. Let $\mathscr{S}$ denote the semigroup of matrices generated by $\mathscr{A}$. By assumption the elements of $\mathscr{S}$ span $M_d(\mathbb{C})$ as a vector space and so there exist matrices $B_1, \ldots, B_{d^2} \in \mathscr{S}$ which form a basis for $M_d(\mathbb{C})$. The eigenvalues of every element of $\mathscr{S}$ are inside the closed unit disc. Hence

$$|\mathrm{Tr}(B)| = |B(1,1) + B(2,2) + \cdots + B(d,d)| \leqslant d \quad \text{for all } B \in \mathscr{S}.$$

Let $X \in \mathscr{S}$ and let $x_{i,j}$ denote the (i,j)-entry of X; i.e., $x_{i,j} = X(i,j)$. Let $\theta_k = \mathrm{Tr}(XB_k)$ for $1 \leqslant k \leqslant d^2$. Then

$$|\theta_k| \leqslant d \quad \text{for } 1 \leqslant k \leqslant d^2. \tag{2.1}$$

We now use an idea which goes back to Burnside of setting up a system of equations and showing we have a unique solution. We have d^2 linear equations in d^2 variables; namely,

$$\sum_{i=1}^{d} \sum_{j=1}^{d} x_{i,j} B_k(j,i) = \theta_k$$

for $1 \leqslant k \leqslant d^2$. Moreover, these equations have a unique solution since $B_1, \ldots, B_{d^2}$ are linearly independent. We can express this equation as

$$C\mathbf{x} = \mathbf{y},$$

where $\mathbf{x}$ is a $d^2 \times 1$ vector whose coordinates are given by the $x_{i,j}$, C is an invertible $d^2 \times d^2$ matrix whose entries are given by the entries of $B_1, \ldots, B_{d^2}$, and $\mathbf{y}$ is a $d^2 \times 1$ vector whose entries are given by $\theta_1, \ldots, \theta_{d^2}$. By Cramer's rule, we can express each $x_{i,j}$ as a ratio of two determinants, in which the denominator is simply the determinant of C and the numerator is the determinant of a matrix obtained by replacing one of the columns of C with $\mathbf{y}$. Using cofactor expansion it follows that each $x_{i,j}$ has an expression of the form

$$(\det(C))^{-1} \sum_{k=1}^{d^2} \theta_k C_k,$$

where C_k is some $(d^2-1) \times (d^2-1)$ minor of C up to sign. Pick $c > 0$ larger than the absolute value of every $(d^2-1) \times (d^2-1)$ minor of C. Then we see that

$$\begin{aligned} |x_{i,j}| &\leqslant \left| (\det(C))^{-1} \sum_{k=1}^{d^2} \theta_k C_k \right| \\ &\leqslant |\det(C)|^{-1} \sum_{k=1}^{d^2} c|\theta_k| \end{aligned}$$

$$\leqslant |\det(C)|^{-1}\sum_{k=1}^{d^2} cd$$
$$\leqslant cd^3|\det(C)|^{-1}.$$

Thus, there is some constant $c' > 0$ such that for any $B \in \mathscr{S}$, the entries of B are at most c'. Hence for $B \in \mathscr{S}$,

$$\begin{aligned}\|B\| &\leqslant \sum_{1\leqslant i,j\leqslant d} \|B(i,j)e_{i,j}\| \\ &\leqslant \sum_{1\leqslant i,j\leqslant d} |B(i,j)|\cdot\|e_{i,j}\| \\ &\leqslant c' \sum_{1\leqslant i,j\leqslant d} \|e_{i,j}\|,\end{aligned}$$

where $e_{i,j}$ is the $d \times d$ matrix with a 1 in the i, j entry and all other entries equal to 0. Thus, the norms of the elements of $\mathscr{S}$ are uniformly bounded. Consequently, $m_n(\mathscr{A})$ is uniformly bounded. □

We now look at finitely generated semigroups which generate a ring which is semisimple.

Proposition 2.5. *Let $\mathscr{A} = \{A_1, \ldots, A_e\}$ be a collection of $d \times d$ complex matrices which generate a semisimple subalgebra of $M_n(\mathbb{C})$ as a $\mathbb{C}$-algebra; moreover suppose that the semigroup generated by $\mathscr{A}$ has the property that every element of this semigroup has all of its eigenvalues on or inside the unit disc. Then $m_n(\mathscr{A})$ is uniformly bounded.*

Proof. We regard $M_d(\mathbb{C})$ as the ring of endomorphisms of a d-dimensional vector space V. Let R denote the $\mathbb{C}$-algebra generated by $\mathscr{A}$. Since R is semisimple, V decomposes into a direct sum of simple R-modules. Write this decomposition as

$$V = \bigoplus_{i=1}^{m} V_i.$$

Our matrix norm on $M_d(\mathbb{C})$ induces a matrix norm, $\|\cdot\|_i$ on $\mathrm{End}(V_i)$, as follows. Given $f \in \mathrm{End}(V_i)$, we take the unique matrix $B_f \in M_d(\mathbb{C})$ which acts as the zero operator on V_j for $j \neq i$ and acts as f on V_i. We then define $\|f\|_i = \|B_f\|$. Notice that the operators $\{A_1|_{V_i}, \ldots, A_e|_{V_i}\}$ generate $\mathrm{End}(V_i)$ since V_i is a simple R-module. Moreover, these operators generate a semigroup whose every element has all of its eigenvalues on or inside the unit disc. Consequently, there exist constants $c_1, \ldots, c_m > 0$ such that

$$\|B|_{V_i}\|_i < c_i$$

for all $B \in \mathscr{S}$ and $1 \leqslant i \leqslant m$ by Proposition 2.4. By construction, for $B \in \mathscr{S}$,

$$\|B\| \leqslant \sum_{i=1}^{m} \|B|_{V_i}\|_i < c_1 + \cdots + c_m.$$

The result follows. □

We are now ready to prove our main result.

Theorem 2.6. *Let $\mathscr{A} = \{A_1, \ldots, A_e\}$ be a finite set of $d \times d$ complex matrices which generate a semigroup in which every element has all of its eigenvalues on or inside the unit disc. Then $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$.*

We prove this claim by induction on d. Notice that when $d = 1$, the claim is trivially true. Next suppose the claim is true for all $m < d$, $d \geqslant 2$. Let R denote the $\mathbb{C}$-algebra generated by $\mathscr{A}$. Let $J(R)$ denote the Jacobson radical of R. If $J(R) = (0)$ then R is a semisimple ring and the result follows from Proposition 2.5. If $J(R) \neq (0)$, then $J(R)$ is a nilpotent ideal since R is Artinian (see, for example, Theorem 1.3.1 of Herstein [6]). Consequently, there is some nonzero subspace of V which is annihilated by $J(R)$. Let W be the subspace of V consisting of all vectors annihilated by $J(R)$. Then W is a nonzero subspace invariant under R. Pick some subspace W' of V such that $W \oplus W' = V$. Define

$$d_1 = \dim(W_1) \quad \text{and} \quad d_2 = \dim(W_2). \tag{2.2}$$

By conjugating $A_1, \ldots, A_e$ by an appropriate matrix, we may assume that each A_i can be written as

$$A_i = \begin{pmatrix} B_i & N_i \\ 0 & C_i \end{pmatrix},$$

with $B_i \in \mathrm{End}(W)$ and $C_i \in \mathrm{End}(W')$ for $1 \leqslant i \leqslant k$. We define

$$A_i' = \begin{pmatrix} B_i & 0 \\ 0 & C_i \end{pmatrix},$$

$$B_i' = \begin{pmatrix} B_i & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$C_i' = \begin{pmatrix} 0 & 0 \\ 0 & C_i \end{pmatrix}.$$

Finally, we take

$$M_i := A_i - A_i'$$

for $1 \leqslant i \leqslant e$. By assumption the semigroup generated by $A_1, \ldots, A_e$ has the property that every matrix in this semigroup has all of its eigenvalues on or inside the unit disc. Hence the semigroups generated by $\mathscr{B} := \{B'_1, \ldots, B'_e\}$ and $\mathscr{C} := \{C'_1, \ldots, C'_e\}$ must also have this property. Notice that we have norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $M_{d_1}(\mathbb{C})$ and $M_{d_2}(\mathbb{C})$ respectively, given by

$$\|X\|_1 = \left\| \begin{pmatrix} X & 0^{d_1\times d_2} \\ 0^{d_2\times d_1} & 0^{d_2\times d_2} \end{pmatrix} \right\|$$

and

$$\|X\|_2 = \left\| \begin{pmatrix} 0^{d_1\times d_1} & 0^{d_1\times d_2} \\ 0^{d_2\times d_1} & X \end{pmatrix} \right\|.$$

Hence by the inductive hypothesis,

$$m_n(\mathscr{B}) = \mathrm{O}(n^{d_1-1}) \quad \text{and} \quad m_n(\mathscr{C}) = \mathrm{O}(n^{d_2-1}).$$

Thus, there exists some constant $c > 0$ such that

$$m_n(\mathscr{B}) \leqslant cn^{d_1-1} \quad \text{and} \quad m_n(\mathscr{C}) \leqslant cn^{d_2-1}. \tag{2.3}$$

Notice that any word of the form $X_{i_1} \cdots X_{i_n}$ in which $X_j \in \{A'_j, M_j\}$ and in which $X_j = M_j$ for at least two values of j must be 0. Hence

$$\begin{aligned} A_{i_1} \cdots A_{i_n} &= \left(A'_{i_1} + M_{i_1}\right) \cdots \left(A'_{i_n} + M_{i_n}\right) \\ &= A'_{i_1} \cdots A'_{i_n} + \sum_{j=1}^{n} A'_{i_1} \cdots A'_{i_{j-1}} M_{i_j} A'_{i_{j+1}} \cdots A'_{i_n} \\ &= A'_{i_1} \cdots A'_{i_n} + \sum_{j=1}^{n} B'_{i_1} \cdots B'_{i_{j-1}} M_{i_j} C'_{i_{j+1}} \cdots C'_{i_n}. \end{aligned}$$

Hence

$$\begin{aligned} \|A_{i_1} \cdots A_{i_n}\| &\leqslant \left\|A'_{i_1} \cdots A'_{i_n}\right\| + \sum_{j=1}^{n} \left\|B'_{i_1} \cdots B'_{i_{j-1}}\right\| \cdot \|M_{i_j}\| \cdot \left\|C'_{i_{j+1}} \cdots C'_{i_n}\right\| \\ &\leqslant m_n(\mathscr{A}') + \sum_{j=1}^{n} m_{j-1}(\mathscr{B}) m_{n-j}(\mathscr{C}) \|M_{i_j}\| \\ &\leqslant m_n(\mathscr{B}) + m_n(\mathscr{C}) + \sum_{j=1}^{n} \left(c(j-1)^{d_1-1}\right)\left(c(n-j)^{d_2-1}\right) \|M_{i_j}\| \\ &\leqslant cn^{d_1-1} + cn^{d_2-1} + \left(\max_{1\leqslant i\leqslant e} \|M_i\|\right) \sum_{j=1}^{n} (cn^{d_1-1})(cn^{d_2-1}) \\ &\leqslant cn^{d_1-1} + cn^{d_2-1} + n\left(\max_{1\leqslant i\leqslant e} \|M_i\|\right) c^2 n^{d_1+d_2-2} \\ &= \mathrm{O}(n^{d-1}). \end{aligned}$$

The result follows. □

Corollary 2.7. *Let $\mathscr{A}$ be a finite set of complex $d \times d$ matrices. Then either there is some $c > 0$ such that $m_n(\mathscr{A}) > c^n$ for all n sufficiently large, or $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$. Moreover, $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$ if and only if every matrix in the semigroup generated by $\mathscr{A}$ has all of its eigenvalues on or inside the unit circle.*

Proof. Let $\mathscr{S}$ be the semigroup generated by $\mathscr{A}$. If $\mathscr{S}$ has a matrix B with an eigenvalue λ with $|\lambda| > 1$, then there is some $c > 1$ such that $m_n(\mathscr{S}) > c^n$ for all n sufficiently large by Lemma 2.3. If, on the other hand, every matrix in $\mathscr{S}$ has all of its eigenvalues on or inside the unit circle, then $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$ by Theorem 2.6. The result follows. $\square$

The following example shows that $\mathrm{O}(n^{d-1})$ in the statement of Corollary 2.7 cannot be replaced by $\mathrm{O}(n^{\alpha})$ with $\alpha < d - 1$.

Example 2.8. Let $\|\cdot\|$ be the sup norm and let $\mathscr{A} = \{A\}$ with $A = \mathrm{I}_d + N$, where N is the $d \times d$ matrix with 1's along the superdiagonal and 0's everywhere else. Then $m_n(\mathscr{A}) \geqslant \binom{n}{d-1} \sim n^{d-1}/(d-1)!$ and $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$.

Proof. Notice that

$$A^n = \mathrm{I}_d + nN + \binom{n}{2}N^2 + \cdots + \binom{n}{d-1}N^{d-1}.$$

Let $\mathbf{v} = (0, 0, \ldots, 1)^{\mathrm{T}}$. Then

$$A^n\mathbf{v} = \begin{pmatrix} \binom{n}{d-1} \\ \binom{n}{d-2} \\ \vdots \\ 1 \end{pmatrix}.$$

Thus, $\|A^n\| \geqslant \binom{n}{d-1}$. Hence, $m_n(\mathscr{A}) \geqslant \binom{n}{d-1} \sim n^{d-1}/(d-1)!$ Notice every eigenvalue of A^n is 1 and so we see that $m_n(\mathscr{A}) = \mathrm{O}(n^{d-1})$ by Corollary 2.7. $\square$

We remark that if one looks at a finite set $\mathscr{A}$ of matrices with entries which are algebraic numbers and ask whether or not $m_n(\mathscr{A})$ is polynomially bounded, then an effective decision procedure for answering this question does not exist. This follows from the fact that $m_n(\mathscr{A})$ is polynomially bounded if and only if the joint spectral radius of $\mathscr{A}$ is bounded by 1 by Corollary 2.7, which is known to be undecidable for matrices with algebraic number entries by Theorem 1 of Blondel and Tsitsiklis [2] (see also comments on page 72 of Daubechies and Lagarias [4]).

Finally we remark that, although probably unrelated, there are other curious gap results for semigroups of matrices with a similar flavour to Corollary 2.7. First, Okniński and Salwa's analogue of the Tits alternative [9] gives a dichotomy result about finitely generated semigroups of invertible matrices which states that either such a semigroup contains a free semigroup on two generators or it generates a nil-potent-by-finite group. Given a finitely generated semigroup $\mathscr{S}$ with finite generating set $\mathscr{A}$, we define the *growth function* of $\mathscr{S}$ with respect to $\mathscr{A}$ to be the function whose value at n is the number of inequivalent words of length n in $\mathscr{A}$. If $\mathscr{S}$ contains a free semigroup on two generators, then the growth function of $\mathscr{S}$ grows exponentially with respect to any generating set. On the other hand, if $\mathscr{S}$ is a subsemigroup of a nilpotent-by-finite group, then the growth function of $\mathscr{S}$ is polynomially bounded with respect to any generating set by a theorem of Bass (see Theorem 11.14 of Krause and Lenagan [7]). Thus growth functions give another example of a gap between polynomially bounded growth and exponential growth for semigroups of matrix rings.

Another example of this type of gap result comes from *subgroup growth*. If G is a finitely generated group, we define the *subgroup growth function* to be the function whose value at n is the number of subgroups of index at most n. The Lubotzky alternative, see [8], allows one to deduce that if G is a finitely generated subgroup of the invertible matrices over a field of characteristic 0, then either G has polynomially bounded subgroup growth or the number of subgroup growth of index at most n is at least $\exp(b(\log n)^2/\log\log n)$ for some $b > 0$ and all n sufficiently large [8]. This gives another example of a gap in possible growths for a function which measures, in some way, the growth of a semigroup of a matrix ring. We see, therefore, that "gaps" of the type in Corollary 2.7 occur in many different settings for semigroups of matrix rings.

Acknowledgment

I thank Jeffrey Lagarias and the referee for many helpful comments and for pointing out some references on semigroups of matrices.

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