A CORRIGENDUM TO “D-GROUPS AND THE DME”

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The purpose of this note, which is not intended for publication, is to fill a small gap that appears in the proof of Proposition 3.5 of [1]. At the bottom of page 364 there is a mistake: when addressing the case of $n = 1$; that is, the case when a and $\log \delta a := \frac{\delta a}{a}$ are algebraically dependent over k; it is claimed that something forces the polynomial $P_0(x)$ to be trivial. In fact, $P_0(x) = d(1 - x)$ for any $d \in k$ is possible, and should have been dealt with.

Instead of following the details in [1] we will give a direct and more conceptual proof of Proposition 3.5 of [1] in this special case:

Proposition. *Suppose k is a field of characteristic zero, R is a commutative affine Hopf k-algebra, and δ is a k-linear derivation on R that is an a-coderivation for some group-like $a \in R$. If a and $\log \delta a$ are algebraically dependent over k then $\log \delta a = d(1 - a)$ for some $d \in k$.*

In particular, the conclusion of [1, Proposition 3.5] *holds with $c := -\frac{d^2}{2}$.*

Proof. There is an affine algebraic group G, a nontrivial character $a : G \to \mathbb{G}_m$, and an a-twisted D-group structure $s : G \to TG$, all over k, such that $R = k[G]$ and δ is the derivation on $k[G]$ induced by s.

Consider the map $\pi := (a, \log \delta a) : G \to \mathbb{G}_m \ltimes \mathbb{G}_a$ that appears in [1] where it is shown, by an easy computation at the beginning of the proof of Proposition 3.8, that, since a is a character and s is a-twisted, π is a morphism of algebraic groups. Since $\{a, \log \delta a\}$ is algebraically dependent over k, and $a \neq 1$, the (connected) algebraic subgroup $H := \pi(G) \leq G_m \ltimes \mathbb{G}_a$ must be 1-dimensional. Hence the co-ordinate projection $\pi_1 : H \to \mathbb{G}_m$ is surjective with finite kernel. It follows that π_1 factors as $\rho_n \phi$, for some $n > 0$, where $\phi : H \to \mathbb{G}_m$ is an isomorphism of algebraic groups and $\rho_n : \mathbb{G}_m \to \mathbb{G}_m$ is given by $\rho_n(x) = x^n$. Let $\mathbb{G}_m \ltimes_n \mathbb{G}_a$ be the semidirect product where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ by $(x, y) \mapsto x^n y$.

We first claim that $F : H \to \mathbb{G}_m \ltimes_n \mathbb{G}_a$ given by $(x, y) \mapsto \big(\phi(x, y), y\big)$, is a group homomorphism. Indeed,

$$\begin{aligned}
F\big((x,y)(x',y')\big) &= F(xx', y + xy') \\
&= \big(\phi(xx', y + xy'), y + xy'\big) \\
&= \big(\phi(x,y)\phi(x',y'), y + xy'\big) \quad \text{as } \phi \text{ is a group homomorphism} \\
&= \big(\phi(x,y)\phi(x',y'), y + \phi(x,y)^n y'\big) \qquad \text{as } \phi(x,y)^n = x \\
&= \big(\phi(x,y), y\big)\big(\phi(x',y'), y'\big) \\
&= F(x,y)F(x',y')
\end{aligned}$$

as desired.

Next, we claim that $\chi := \pi_2 \phi^{-1} : \mathbb{G}_m \to \mathbb{G}_a$ is a ρ_n-twisted additive character. That is, that $\chi(xx') = \chi(x) + x^n \chi(x')$. Indeed, note that the image $H' = F(H)$ is

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the graph of χ. So for $x, x' \in \mathbb{G}_m$ we have $\big(xx', \chi(xx')\big) \in H'$. But as F is a group homomorphism, H' is a subgroup of $\mathbb{G}_m \ltimes_n \mathbb{G}_a$, and so

$$\big(x, \chi(x)\big)\big(x', \chi(x')\big) = \big(xx', \chi(x) + x^n\chi(x')\big) \in H',$$

as well. It follows that $\chi(xx') = \chi(x) + x^n\chi(x')$, as desired.

Choose $b \in k$ such that $b^n \neq 1$. We have, for $x \in \mathbb{G}_m$,

$$\begin{array}{rcl} \chi(xb) & = & \chi(x) + x^n\chi(b) \quad \text{and} \\ \chi(bx) & = & \chi(b) + b^n\chi(x) \end{array}$$

so that $\chi(x)(1-b^n) = \chi(b)(1-x^n)$. Letting $d := \frac{\chi(b)}{1-b^n}$ we get $\chi(x) = d(1-x^n)$.

So, for all $g \in G$, we have

$$\begin{array}{rcll} \log\delta a(g) & = & \pi_2\pi(g) & \\ & = & \chi\phi\pi(g) & \text{as } \chi = \pi_2\phi^{-1} \\ & = & d\big(1 - \phi\pi(g)^n\big) & \\ & = & d\big(1 - \pi_1\pi(g)\big) & \text{as } \pi_1 = \rho_n\phi \\ & = & d\big(1 - a(g)\big). & \end{array}$$

That is, $\log\delta a = d(1-a)$, as desired.

For the "in particular" clause, a direct computation shows that for $c := -\frac{d^2}{2}$ we have the identity $a\delta^2 a = \frac{3}{2}(\delta a)^2 + c(a^2 - a^4)$ as claimed in Proposition 3.5 of [1]. $\square$

It may be worth pointing out that what $\log\delta a = d(1-a)$ says geometrically is that we have the short exact sequence

$$1 \longrightarrow \big(\ker(a), u\big) \longrightarrow (G, s) \xrightarrow{\ a\ } (\mathbb{G}_m, t_d) \longrightarrow 1$$

where $u := s\restriction_{\ker(a)}$ makes $\big(\ker(a), u\big)$ a D-group, $t_d(x) := d(x - x^2)$ makes $(\mathbb{G}_m, t_d)$ an id-twisted D-group, and the morphisms are algebraic group homomorphisms that are also morphisms of D-varieties.

References

[1] Jason Bell, Omar León Sánchez, and Rahim Moosa. *D*-groups and the Dixmier-Moeglin equivalence. *Algebra Number Theory*, 12(2):343–378, 2018.