

## CENTRALIZERS IN DOMAINS OF GELFAND–KIRILLOV DIMENSION 2

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### ABSTRACT

Given an affine domain of Gelfand–Kirillov dimension 2 over an algebraically closed field, it is shown that the centralizer of any non-scalar element of this domain is a commutative domain of Gelfand–Kirillov dimension 1 whenever the domain is not polynomial identity. It is shown that the maximal subfields of the quotient division ring of a finitely graded Goldie algebra of Gelfand–Kirillov dimension 2 over a field  $F$  all have transcendence degree 1 over  $F$ . Finally, centralizers of elements in a finitely graded Goldie domain of Gelfand–Kirillov dimension 2 over an algebraically closed field are considered. In this case, it is shown that the centralizer of a non-scalar element is an affine commutative domain of Gelfand–Kirillov dimension 1.

### 1. Introduction

In this paper we consider domains of Gelfand–Kirillov dimension 2. This dimension was first considered by Gelfand and Kirillov in 1966 [6]. It is defined as follows. Given a field  $F$  and an affine  $F$ -algebra  $A$ , we define the *Gelfand–Kirillov dimension* of  $A$  to be

$$\text{GKdim}(A) := \limsup_{n \rightarrow \infty} \log(\dim V^n) / \log n,$$

where  $V$  is a finite-dimensional subspace of  $A$  that generates  $A$  as an  $F$ -algebra. We note that this definition is independent of choice of  $V$ . In the case that  $A$  is not affine, we define the Gelfand–Kirillov dimension of  $A$  to be the supremum of the Gelfand–Kirillov dimensions of all affine subalgebras of  $A$ . Basic facts about Gelfand–Kirillov dimension can be found in [6].

An affine domain of Gelfand–Kirillov dimension 0 is a division ring that is finite-dimensional over its center. An affine domain of Gelfand–Kirillov dimension 1 over a field  $F$  is a finite module over its center, and hence polynomial identity (see [9]); if, in addition,  $F$  is algebraically closed, a routine application of Tsen’s theorem shows that this domain is in fact commutative. Domains of Gelfand–Kirillov dimension 2 are not well understood. The second author conjectures that such domains must either be primitive or polynomial identity. In the case that such a domain is a finitely generated  $\mathbb{N}$ -graded algebra with the property that the homogenous elements of any given degree form a finite-dimensional vector space, it has been shown by Artin and Stafford [2] that this conjecture holds. We consider the centralizers of elements of an affine domain of Gelfand–Kirillov dimension 2. Centralizers of the first Weyl algebra – a domain of Gelfand–Kirillov dimension 2 – were studied as long ago as 1922 by Burchnell and Chaundy [4], and also by Amitsur [1]. Using the work of Artin and Stafford, we are also able to obtain results about the centralizers of

elements of the quotient division algebra of a finitely graded domain of Gelfand–Kirillov dimension 2. We state our main results.

**THEOREM 1.1.** *Let  $A$  be a non-polynomial identity affine domain of Gelfand–Kirillov dimension 2 over an algebraically closed field  $F$ . Then the centralizer of a non-scalar element of  $A$  is a commutative domain of Gelfand–Kirillov dimension 1.*

In the case where we are dealing with graded rings, we obtain somewhat stronger results. We first give the following definition.

**DEFINITION 1.2.** Let

$$A = \bigoplus_{n=0}^{\infty} A_n$$

be a finitely generated  $\mathbb{N}$ -graded  $F$ -algebra. We say that  $A$  is *finitely graded* if  $\dim_F(A_n) < \infty$  for all  $n \geq 0$ .

**THEOREM 1.3.** *Let  $A$  be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension 2 over a field  $F$ . Then any subfield of  $Q(A)$  has transcendence degree at most 1 over  $F$ .*

**THEOREM 1.4.** *Let  $A$  be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension 2 over an algebraically closed field  $F$ . Then the centralizer of a non-scalar element  $a \in Q(A)$  is a finitely generated field extension of  $F$  of transcendence degree 1.*

## 2. Algebras of Gelfand–Kirillov dimension 2

**LEMMA 2.1.** *Let  $A$  be a non-polynomial identity affine domain of Gelfand–Kirillov dimension 2 over an algebraically closed field  $F$ , and let  $B$  be a polynomial identity subalgebra of  $A$ . Then  $B$  has Gelfand–Kirillov dimension at most 1.*

*Proof.* Suppose that  $B$  has Gelfand–Kirillov dimension greater than 1. Then  $\text{GKdim}(B) > \text{GKdim}(A) - 1$ . By a theorem of Borho and Kraft (see [3]),  $Q(A)$  is a finite-dimensional vector space over  $Q(B)$ . In particular,  $Q(A)$  is polynomial identity, since  $Q(B)$  is polynomial identity, a contradiction. It follows that  $B$  must have Gelfand–Kirillov dimension 1.  $\square$

**THEOREM 2.2.** *Let  $A$  be a non-polynomial identity affine domain of Gelfand–Kirillov dimension 2 over an algebraically closed field  $F$ . Then the centralizer of a non-scalar element of  $A$  is a commutative domain of Gelfand–Kirillov dimension 1.*

*Proof.* Let  $a$  be a non-scalar element of  $A$ . Let  $C(a)$  denote the centralizer of  $a$  in  $A$ , and let  $B$  be an affine subalgebra of  $C(a)$  containing  $a$ . Notice that the Gelfand–Kirillov dimension of  $B$  is at most 2, as it is a subalgebra of  $A$ . On the other hand, the center of  $B$  has Gelfand–Kirillov dimension at least 1, since it contains  $F[a]$ . By [10, Corollary 2],  $B$  is polynomial identity. By Lemma 2.1,  $B$  has Gelfand–Kirillov

dimension 1. By [9],  $B$  is commutative. Since  $B$  is an arbitrary affine subalgebra of  $C(a)$ , we see that  $C(a)$  is a commutative domain of Gelfand–Kirillov dimension 1. □

### 3. Graded algebras of Gelfand–Kirillov dimension 2

In this section we prove our main results for graded domains of Gelfand–Kirillov dimension 2. Throughout this section we use arguments involving Krull dimension.

NOTATION 3.1. Given an algebra  $A$  and a right  $A$ -module  $M$ , we denote the Krull dimension of  $M$  by  $\mathcal{K}(M)$ .

For basic facts about Krull dimension we refer the reader to [7].

LEMMA 3.2. *Let  $A$  be a prime Noetherian polynomial identity ring that is not primitive. Suppose that  $\sigma$  is an automorphism of  $A$  such that  $A[x, x^{-1}; \sigma]$  is simple. Then*

$$\mathcal{K}(A) = \mathcal{K}(A[x, x^{-1}; \sigma]).$$

*Proof.* The proof is similar to that of [7, Corollary 6.6.7]. We show that there are no nonzero right  $A[x, x^{-1}; \sigma]$ -modules of finite length over  $A$ . Suppose that  $M$  is a nonzero right  $A[x, x^{-1}; \sigma]$ -module of finite length over  $A$ . Since  $A$  is not primitive,  $\text{ann}_A(M)$  must contain a finite product of primitive ideals of  $A$ . Thus  $A/\text{ann}_A(M)$  is a subdirect product of semisimple Artinian rings. It follows that  $\mathcal{K}(A/\text{ann}_A(M)) = 0$ . Notice that  $\text{ann}_A(M)$  is stable under  $\sigma$ , and hence generates a proper nonzero two-sided ideal of  $A[x, x^{-1}; \sigma]$ , contradicting the fact that  $A[x, x^{-1}; \sigma]$  is simple. The result now follows from [7, Theorem 6.6.10]. □

We now prove a useful lemma about subfields of quotient division algebras.

LEMMA 3.3. *Let  $A$  be a finitely graded Goldie domain of Gelfand–Kirillov dimension 2. Then the subfields of  $Q(A)$  are finitely generated.*

*Proof.* By [2, Theorem 0.1] we know that  $A$  has a graded quotient ring

$$Q_{\text{gr}}(A) \cong D[x, x^{-1}; \sigma]$$

for some division ring  $D$  which is a finite module over its center and some automorphism  $\sigma$  of  $D$ . By [8, Theorem 3] the subfields of  $Q(D[x, x^{-1}; \sigma])$  are finitely generated. The result follows. □

PROPOSITION 3.4. *Let  $A$  be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension 2 over a field  $F$ . Then  $Q_{\text{gr}}(A)$  is a simple  $F$ -algebra and  $Z(Q_{\text{gr}}(A))$  is a finite extension of  $F$  consisting of homogeneous elements of degree 0.*

*Proof.* Note that since  $A$  is a finitely generated domain of Gelfand–Kirillov dimension 2, we have

$$\text{GKdim}(A/I) \leq 1$$

for any nonzero ideal  $I$  of  $A$ . In particular,  $A/I$  is polynomial identity, by [9]. It follows that  $A/I$  satisfies a multilinear homogeneous polynomial identity, say  $p(x_1, \dots, x_d)$ . Now since  $p$  is multilinear and  $A$  is not polynomial identity, there exist homogeneous elements  $a_1, \dots, a_d \in A$  such that  $u := p(a_1, \dots, a_d) \neq 0$ . Thus  $u$  is a nonzero homogeneous element of  $A$  which is in  $I$ . It follows that every nonzero ideal of  $I$  contains a homogeneous regular element  $u$ . Thus  $Q_{\text{gr}}(A)$  is simple. By [2, Theorem 0.1],

$$Q_{\text{gr}}(A) \cong D[x, x^{-1}; \sigma]$$

with  $D$  finite-dimensional over its center  $Z$  and  $Z$  a finitely generated field extension of transcendence degree 1 over  $F$ . Note that if  $a \in Q_{\text{gr}}(A)$  is central, then each of the homogeneous parts of  $a$  must also be central. Suppose that  $b = \alpha x^i$  with  $\alpha \in D$  and  $i \neq 0$  is central. Then since  $D$  is a finite module over its center,

$$D[\alpha x^i, (\alpha x^i)^{-1}] = D[x^i, x^{-i}; \sigma^i]$$

is polynomial identity. Since  $D[x, x^{-1}; \sigma]$  is a finite module over  $D[x^i, x^{-i}; \sigma^i]$ , we conclude that  $Q_{\text{gr}}(A)$  is polynomial identity, a contradiction. Thus any homogeneous central element of  $Q_{\text{gr}}(A)$  must have degree 0. Consequently,  $Z(Q_{\text{gr}}(A)) \subseteq D$ . Next, suppose that  $\alpha \in D$  is in  $Z(Q_{\text{gr}}(A))$ . Then  $F(\alpha)[x, x^{-1}]$  is commutative. We claim that  $\alpha$  is algebraic over  $F$ . To see this, suppose that this is not the case. Then  $[Z : F(\alpha)] < \infty$ , and hence  $Q_{\text{gr}}(A)$  is a finite module over  $F(\alpha)[x, x^{-1}]$ . Since  $F(\alpha)[x, x^{-1}]$  is commutative, we conclude that  $Q_{\text{gr}}(A)$  is polynomial identity, which contradicts the fact that  $A$  is not polynomial identity. Thus  $\alpha$  is algebraic over  $F$ . It follows that  $Z(Q_{\text{gr}}(A))$  is an algebraic extension of  $F$ . By Lemma 3.3, it is also a finitely generated extension. The result follows.  $\square$

**THEOREM 3.5.** *Let  $A$  be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension 2 over a field  $F$ . Then any subfield of  $Q(A)$  has transcendence degree at most 1 over  $F$ .*

*Proof.* By [2, Theorem 0.1],  $A$  has a graded quotient

$$Q_{\text{gr}}(A) = D[x, x^{-1}; \sigma],$$

where  $D$  is a division algebra which is finite-dimensional over its center  $Z$ , with  $Z$  a finitely generated extension of  $F$  of transcendence degree 1. Let  $K$  be a subfield of  $Q_{\text{gr}}(A)$  that is a purely transcendental extension of  $F' := Z(Q_{\text{gr}}(A)) \subseteq D$ . Observe that  $K \otimes_{F'} Q_{\text{gr}}(A) \cong (D \otimes_{F'} K)[x, x^{-1}; \sigma]$ . Now  $D \otimes_{F'} K$  is a prime Noetherian algebra. Since it is polynomial identity and not simple, it is not primitive. Note that

$$Q_{\text{gr}}(A) \otimes_{F'} K \cong (D \otimes_{F'} K)[x, x^{-1}; \sigma]$$

is simple, because by Proposition 3.4,  $Q_{\text{gr}}(A)$  is a central simple  $F'$ -algebra and  $K$  is a simple  $F'$ -algebra. By Lemma 3.2,

$$\mathcal{K}(Q_{\text{gr}}(A) \otimes_{F'} K) = \mathcal{K}(D \otimes_{F'} K).$$

Since  $D$  is a finite  $Z$ -module, we have

$$\mathcal{K}(D \otimes_{F'} K) = \mathcal{K}(Z \otimes_{F'} K) = \min\{\text{trdeg}_{F'}(Z), \text{trdeg}_{F'}(K)\},$$

and so

$$\mathcal{K}(Q_{\text{gr}}(A) \otimes_{F'} K) = \min\{\text{trdeg}_{F'}(Z), \text{trdeg}_{F'}(K)\}. \tag{3.1}$$

We also have

$$\text{trdeg}_{F'}(K) = \mathcal{K}(K \otimes_{F'} K).$$

By [7, Corollary 6.5.3], we have  $\mathcal{K}(K \otimes_{F'} K) \leq \mathcal{K}(Q(A) \otimes_{F'} K)$ . Additionally, by [7, Lemma 6.5.3.ii], we have

$$\mathcal{K}(Q(A) \otimes_{F'} K) \leq \mathcal{K}(Q_{\text{gr}}(A) \otimes_{F'} K) = \min(\text{trdeg}_{F'}(K), \text{trdeg}_{F'}(Z)).$$

Since  $Z$  has transcendence degree at most 1 over  $F'$ , we see that  $\text{trdeg}_{F'}(K) \leq 1$ . Since  $F'$  is a finite extension of  $F$ , we obtain the desired result. □

We are now ready to prove our main result for graded domains of Gelfand–Kirillov dimension 2.

**THEOREM 3.6.** *Let  $A$  be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension 2 over an algebraically closed field  $F$ . Then the centralizer of a non-scalar element  $a \in Q(A)$  is a finitely generated field extension of  $F$  of transcendence degree 1.*

*Proof.* Observe that  $C(a)$  is a division algebra over  $F$ . Notice also that for any  $b \in C(a)$ ,  $a$  and  $b$  commute and hence  $F(a, b)$  is a field. By Theorem 3.5,  $F(a, b)$  has transcendence degree at most 1 over  $F$ . Thus  $b$  is algebraic over  $F(a)$ , as  $F(a)$  has transcendence degree 1 over  $F$ . It follows that  $C(a)$  is a division algebra that is algebraic over  $F(a)$ . Let  $Z$  denote the center of  $C(a)$ . Then  $Z \supseteq F(a)$ , and hence  $Z$  has transcendence degree at least 1 over  $F$ . On the other hand,  $Z$  is a subfield of  $Q(A)$ , and hence has transcendence degree at most 1 over  $F$ . By Lemma 3.3,  $Z$  is a finitely generated field extension of  $F$ . Let  $K$  denote a maximal subfield of  $C(a)$ . Then  $K$  has transcendence degree 1 over  $F$  and is a finitely generated extension of  $F$  by the same lemma. It follows that  $K$  is a finitely generated algebraic extension of  $Z$ , and so  $[K : Z] < \infty$ . It follows that  $C(a)$  is finite-dimensional over  $Z$  using the results from [5, p. 165]; in fact,  $[C(a) : Z] = [K : Z]^2$ . Since  $Z$  is a finitely generated extension of transcendence degree 1 of an algebraically closed field,  $Z$  is necessarily a  $C_1$  field. By Tsen’s theorem,  $C(a)$  is commutative, and hence is a field of transcendence degree 1 over  $F$ . □

**PROPOSITION 3.7.** *Let  $B$  be a commutative graded domain of Gelfand–Kirillov dimension 1 over an algebraically closed field  $F$ . Then there exist positive integers  $m_1, \dots, m_\ell$  and there exists a homogeneous element  $t \in Q_{\text{gr}}(B)$  such that  $B = F[t^{m_1}, \dots, t^{m_\ell}]$ .*

*Proof.* Let

$$\mathcal{S} = \{\text{deg}(u) \mid u \in B, u \text{ homogeneous}\}.$$

There exists some positive integer  $N$  such that every element of  $\mathcal{S}$  can be expressed as a nonnegative integer linear combination of elements of  $\mathcal{S} \cap \{1, 2, \dots, N\}$ . We can choose homogeneous elements  $u_1, \dots, u_\ell \in B$  such that

$$\{\text{deg}(u_1), \dots, \text{deg}(u_\ell)\} = \mathcal{S} \cap \{1, 2, \dots, N\}.$$

Let  $u \in B$  be homogeneous. Now  $R := F[u_1, \dots, u_\ell, u]$  is a finitely graded commutative domain of Gelfand–Kirillov dimension 1. Since  $F$  is algebraically

closed, it follows that  $Q_{\text{gr}}(R) \cong F[t, t^{-1}]$  where  $t$  is homogeneous of degree  $\gcd(\deg(u_1), \dots, \deg(u_\ell))$ . Since  $u_i$  is homogeneous and in  $F[t, t^{-1}]$ , it follows that  $u_i = \alpha_i t^{m_i}$  with  $m_i = \deg(u_i)/\deg(t)$  and  $\alpha_i \in F$ . Similarly,  $u = \alpha t^m$  with  $m = \deg(u)/\deg(t)$ . By construction,  $m$  is a nonnegative integer linear combination of  $m_1, \dots, m_\ell$ . It follows that  $u \in F[u_1, \dots, u_\ell] = F[t^{m_1}, \dots, t^{m_\ell}]$ . Thus

$$B = F[t^{m_1}, \dots, t^{m_\ell}]. \quad \square$$

**THEOREM 3.8.** *Let  $A$  be a finitely graded Goldie  $F$ -algebra with the property that for any homogeneous element of positive degree,  $x \in A$ , there exist positive integers  $m_1, \dots, m_\ell$  and a homogeneous element  $t \in Q_{\text{gr}}(A)$  of positive degree  $d$  such that*

$$C(x) = F[t^{m_1}, \dots, t^{m_\ell}].$$

*Then the centralizer of any element of  $A$  is an affine  $F$ -algebra.*

*Proof.* Let  $a \in A$ . Write  $a = a_0 + a_1 + \dots + a_n$ , where  $a_i$  is homogeneous of degree  $i$  and  $a_n \neq 0$ . Without loss of generality,  $n > 0$ . Observe that if  $b = b_0 + \dots + b_p \in C(a)$  with  $b_i$  homogeneous of degree  $i$ , then  $b_p \in C(a_n)$ . There exist a homogeneous element  $t \in Q_{\text{gr}}(A)$  of degree  $d$  and positive integers  $m_1, \dots, m_\ell$  such that  $C(a_n) = F[t^{m_1}, \dots, t^{m_\ell}]$ . It follows that if  $b = b_0 + \dots + b_p \in C(a)$ , then  $b_p = \alpha t^m$ , with  $\alpha \in F$  and  $m \in \mathbb{N}m_1 + \dots + \mathbb{N}m_\ell$ . Let

$$\mathcal{S} = \{\deg(b) \mid b \in C(a), b \neq 0\}.$$

Let  $d'$  be the greatest common divisor of  $\mathcal{S}$ . Then there exists an integer  $N$  such that all integers larger than  $N$  can be expressed as a nonnegative integer linear combination of elements of  $\mathcal{S}$ . Choose elements  $r_1, \dots, r_\ell \in C(a)$  such that

$$\{\deg(r_1), \dots, \deg(r_\ell)\} = \mathcal{S} \cap \{0, 1, \dots, N\}.$$

Since the leading homogeneous part of  $r_i$  is in  $C(a_n)$ , we can multiply by appropriate scalars so that the leading homogeneous part of  $r_i$  is  $t^{m_i}$  for some positive integer  $m_i$ . We claim that  $C(a) = F[r_1, \dots, r_\ell]$ . To see this, suppose this is not the case. Choose  $b = b_0 + \dots + b_p \in C(a) \setminus F[r_1, \dots, r_\ell]$  with  $p$  minimal. Again, we may assume that  $b_p = t^{p/d}$ . By assumption,  $p/d = i_1 m_1 + \dots + i_\ell m_\ell$  for some nonnegative integers  $i_1, \dots, i_\ell$ . Observe that both  $b$  and  $b' := r_1^{i_1} \dots r_\ell^{i_\ell}$  have degree  $p$  and both have the same homogeneous part of degree  $p$ , namely  $t^{p/d}$ . Thus  $b - b' \in C(a)$  has degree at most  $p - 1$ . By the minimality of  $\deg(b)$ , we see that  $b - b' \in F[r_1, \dots, r_\ell]$ , contradicting the fact that  $b \notin F[r_1, \dots, r_\ell]$ .  $\square$

**COROLLARY 3.9.** *Let  $A$  be a finitely graded non-polynomial identity Goldie domain of Gelfand–Kirillov dimension 2 over an algebraically closed field  $F$ . Then the centralizer of a non-scalar element is an affine commutative domain of Gelfand–Kirillov dimension 1.*

*Proof.* By Theorem 2.2, the centralizer of a non-scalar element  $b \in A$  is a commutative domain of Gelfand–Kirillov dimension 1. Hence by Proposition 3.7 and Theorem 3.8, the centralizer of  $a$  is affine.  $\square$

We make the conjecture that the same result holds in the ungraded case.

CONJECTURE 3.10. Let  $A$  be an affine Noetherian non-polynomial identity domain of Gelfand–Kirillov dimension 2 over an algebraically closed field. Then the centralizer of a non-scalar element is an affine domain.

COROLLARY 3.11. *Let  $A$  be an affine domain of Gelfand–Kirillov dimension 2 with a non-polynomial identity domain for an associated graded ring. Then the centralizer of a non-scalar element is an affine commutative domain of Gelfand–Kirillov dimension 1.*

*Proof.* The same argument used in Theorem 3.8 and Corollary 3.9 gives the result.  $\square$

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