**When structures are almost surely connected**

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**Abstract**

Let $A_n$ denote the number of objects of some type of "size" $n$, and let $C_n$ denote the number of these objects which are connected. It is often the case that there is a relation between a generating function of the $C_n$'s and a generating function of the $A_n$'s. Wright showed that if $\lim_{n\to\infty} C_n/A_n = 1$, then the radius of convergence of these generating functions must be zero. In this paper we prove that if the radius of convergence of the generating functions is zero, then $\limsup_{n\to\infty} C_n/A_n = 1$; moreover, we show that $\liminf_{n\to\infty} C_n/A_n$ can assume any value between 0 and 1.

# 1 Introduction

Let $A_n$ count objects of some type by their "size" $n$ and let $C_n$ count those which are connected. One frequently has either

$$A(x) = \exp(C(x)) \quad \text{or} \quad A(x) = \exp\Big(\sum_{k\geq 1} \frac{C(x^k)}{k}\Big), \tag{1.1}$$

for exponential generating functions of labeled objects and ordinary generating functions of unlabeled objects, respectively. Let $R$ be the radius of convergence of the power series. Various authors have studied the limiting behaviour of $C_n/A_n$. In particular, Wright [3] constructed a sequence $\{C_n\}_{n\geq 1}$ such that $\limsup C_n/A_n = 1$ and $\liminf C_n/A_n < 2/3$ in both the labeled and unlabeled case. Also, Wright [3], [4] showed that if $\lim_{n\to\infty} C_n/A_n = 1$, then $R = 0$. Compton [1] asked if the converse were true, assuming the limit exists. The following theorem provides an affirmative answer.

**Theorem 1** *Suppose that either of (1.1) holds then:*

- *If* $R = 0$*, then* $\limsup_{n\to\infty} C_n/A_n = 1$.

- *For any* $0 \le l \le 1$*, there exists both labeled and unlabeled objects satisfying (1.1) with* $R = 0$ *and* $\liminf_{n\to\infty} C_n/A_n = l$.

Combining the first part of the theorem with Wright's result shows that, if $\lim_{n\to\infty} C_n/A_n = \rho$ exists, then $\rho = 1$ if and only if $R = 0$.

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# 2 Proofs

We require the following simple lemma.

**Lemma 1** *Suppose* $p(x) = \sum_{i=1}^{\infty} p_i x^i$ $(p_1 \neq 0)$ *is analytic at zero and suppose* $h(x) = \sum_{i=1}^{\infty} h_i x^i$ *has the property that* $p(h(x)) = g(x)$ *is a power series that is analytic at zero. Then* $h(x)$ *is analytic at zero.*

**Proof**. Let $p^{-1}(x)$ be the formal inverse of $p$. Since $p(x)$ is analytic at zero, we have that $p^{-1}(x)$ is analytic at zero by [2] page 87, Theorem 4.5.1. Hence $h(x) = p^{-1}(g(x))$ is analytic at zero as required. ∎

We now prove a lemma that will be useful to us.

**Lemma 2** *Suppose* $C(x) = \sum_{i=1}^{\infty} c_i x^i$ *is a power series with non-negative coefficients and*

$$p(x) = \sum_{i=1}^{\infty} p_i x^i \qquad (p_1 \neq 0)$$

*is a power series that is analytic at zero satisfying*

$$p_n + \alpha c_n \le [x^n] e^{C(x)}$$

*for some* $\alpha > 1$ *and all* $n \ge 1$*. Then* $C(x)$ *is analytic at zero.*

**Proof**. To prove this, let us first note that if $D(x) = \sum_{i=1}^{\infty} d_i x^i$ is a formal power series that satisfies the equation

$$p_n + \alpha d_n = [x^n] e^{D(x)} \tag{2.2}$$

for all $n \geq 1$, then $D(x)$ is analytic. To see this, let us note that equation (2.2) is equivalent to stating that

$$1 + p(x) + \alpha D(x) = e^{D(x)} \tag{2.3}$$

as formal power series. Notice that $d_1 = -p_1/(\alpha - 1) \neq 0$ and hence $D(x)$ has a formal inverse $D^{-1}(x)$. Substituting $x = D^{-1}(u)$ into the equation (2.3), we find that

$$p(D^{-1}(u)) = e^u - \alpha u - 1.$$

Thus by Lemma 1 we have that $D^{-1}(u)$ is analytic at zero. By Lemma 1 we have that $D(x)$ is analytic at zero. We now show that $0 \leq c_n \leq d_n$ for all $n \geq 1$. We prove this by induction on $n$. Note that for $n = 1$, we have that $p_1 + \alpha c_1 \leq [x]e^{C(x)} = c_1$ and so $c_1 \leq p_1/(\alpha - 1) = d_1$. Hence the claim is true when $n = 1$. Suppose the claim is true for all values less than $n$. We have

$$\begin{aligned} p_n + \alpha c_n &\leq [x^n]e^{C(x)} \\ &= [x^n]\exp(c_1 x + c_2 x^+ \cdots + c_n x^n) \\ &\leq [x^n]\exp(d_1 x + d_2 x^2 + \cdots + d_n x^n + (c_n - d_n)x^n), \end{aligned}$$

since $c_k \leq d_k$ for $k < n$. Thus

$$\begin{aligned} p_n + \alpha c_n &\leq [x^n]\exp(d_1 x + \cdots + d_n x^n)\exp((c_n - d_n)x^n) \\ &= [x^n]\exp(D(x))(1 + (c_n - d_n)x^n) \\ &= [x^n]\exp(D(x)) + c_n - d_n \\ &= p_n + \alpha d_n + c_n - d_n. \end{aligned}$$

Hence $(\alpha - 1)c_n \leq (\alpha - 1)d_n$ and so $0 \leq c_n \leq d_n$ for all $n \geq 1$. Since $D(x)$ is analytic at zero, it follows that $C(x)$ is analytic at zero. This completes the proof. ∎

The following theorem implies the first part of Theorem 1. To see this, it suffices to note that

$$[x^n]\exp(C(x)) \leq [x^n]\exp\Big(\sum_{k \geq 1} \frac{C(x^k)}{k}\Big).$$

**Theorem 2** *Suppose $c_i \geq 0$ for all $i$ and $C(x) = \sum_{i=1}^{\infty} c_i x^i$ has radius of convergence zero. Let*

$$A(x) = \sum_{i=1}^{\infty} a_i x^i = \exp\Big(\sum_{j=1}^{\infty} C(x^j)/j\Big).$$

*Then*

$$\limsup_{n\to\infty} \frac{c_n}{a_n} = 1.$$

**Proof**. Without loss of generality we may assume that $c_1 \geq 1$, as increasing the value of $c_1$ can only decrease the values of $c_n/a_n$ for large $n$. Suppose

$$\limsup_{n\to\infty} \frac{c_n}{a_n} \neq 1.$$

Then there exists $\lambda > 1$ and a positive integer $N$ such that

$$\frac{a_n}{c_n} > \lambda \quad \text{for all } n > N \tag{2.4}$$

Let $H(x) = \sum_{i=1}^{\infty} h_i x^i$ be the power series

$$H(x) = \sum_{k=1}^{\infty} \frac{C(x^k)}{k} \quad \text{so that} \quad c_n = \sum_{d|n} \frac{\mu(d) h_{n/d}}{d}.$$

Define the two sets

$$S_1 = \left\{ n > N \,\middle|\, \frac{a_n}{h_n} \geq \frac{1+\lambda}{2} \right\} \tag{2.5}$$

and

$$S_2 = \left\{ n > N \,\middle|\, \frac{a_n}{h_n} < \frac{1+\lambda}{2} \right\}. \tag{2.6}$$

If $n \in S_2$, then by (2.4) we must have that $c_n/h_n < (1+\lambda)/2\lambda$. Thus

$$\sum_{d|n} \frac{\mu(d) h_{n/d}}{d} < \frac{(1+\lambda) h_n}{2\lambda}. \tag{2.7}$$

But

$$\begin{aligned} \sum_{d|n} \frac{\mu(d) h_{n/d}}{d} &= h_n + \sum_{\substack{d|n \\ d\neq 1}} \frac{\mu(d) h_{n/d}}{d} \\ &\geq h_n - \sum_{\substack{d|n \\ d\neq 1}} \frac{h_{n/d}}{d}. \end{aligned}$$

Combining this result with (2.7) we find that there exists some divisor $d \neq 1$ of $n$ such that $h_{n/d}/d > (\lambda-1)h_n/2d(n)\lambda$. Hence

$$\begin{aligned} h_n(1+\lambda)/2 > a_n &= [x^n]e^{H(x)} \\ &\geq h_n + h_{n/d}^d/d! \\ &\geq h_n + \frac{((\lambda-1)dh_n)^d}{(2d(n)\lambda)^d d!} \\ &\geq h_n + \frac{(\lambda-1)^d h_n^d}{(2n\lambda)^d}. \end{aligned}$$

Solving for $h_n$ we find that

$$h_n < \left( \frac{2n\lambda \left(\frac{\lambda-1}{2}\right)^{1/d}}{\lambda-1} \right)^{d/(d-1)} = O(n^2)$$

and so there exists $C > 0$ such that $h_n < Cn^2$ for all $n \in S_2 \cup \{1, 2, \ldots, N\}$; that is, all $n \notin S_1$. Define

$$p(x) = -\frac{(1+\lambda)}{2}\Big(\sum_{j=1}^{N} Cj^2x^j + \sum_{j\in S_2} Cj^2x^j\Big).$$

Clearly $p(x)$ has a radius of convergence of at least 1 and so it is analytic at zero. Consider the power series $p(x) + (1+\lambda)H(x)/2$. Notice if $n \notin S_1$, then

$$\begin{aligned} [x^n]\left(p(x) + \frac{(1+\lambda)}{2}H(x)\right) &= \frac{(1+\lambda)}{2}(-Cn^2 + h_n) \\ &\leq 0 \\ &\leq a_n \\ &= [x^n]\exp(H(x)). \end{aligned}$$

If $n \in S_1$, then

$$[x^n]\left(p(x) + \frac{(1+\lambda)}{2}H(x)\right) = (1+\lambda)h_n/2 \leq a_n = [x^n]\exp(H(x)).$$

Hence we have

$$[x^n]\left(p(x) + \frac{(1+\lambda)}{2}H(x)\right) \leq [x^n]\exp(H(x))$$

for all $n \geq 1$. Moreover when $n = 1$, $p'(0) + \frac{1+\lambda}{2} h_1 \leq h_1$, and so $p'(0) < 0$. Hence by Lemma 2, $H(x)$ is analytic at zero. Since $0 \leq c_n \leq h_n$ for all $n$, we see that $C(x)$ is also analytic at zero, a contradiction. This completes the proof of the theorem. ∎

We now prove the second part of Theorem 1. The set of all graphs (labeled or unlabeled) provides an example for $l = 1$ [5]. For $l = 0$, notice if $C(x) = \sum_{n \geq 1} C_n x^n$ is any power series of radius zero having positive integer coefficents and $C_n = 1$ for infinitely many $n$, then in both the labeled and unlabeled cases we have that

$$\begin{aligned} A_n &\geq [x^n] \exp(C(x)) \\ &\geq [x^n] \exp(\frac{x}{1-x}) \\ &\geq [x^n] \frac{1}{2!} \frac{x^2}{(1-x)^2} \\ &= (n-1)/2. \end{aligned}$$

Hence

$$\inf_{\{n \,:\, C_n = 1\}} C_n / A_n = 0.$$

Hence to prove the second part of Theorem 1 it suffices to prove the following theorem.

**Theorem 3** *Given $l$ with $0 < l < 1$, there exist power series $C(x) = \sum_{i \geq 1} c_i x^i$, $H(x) = \sum_{i \geq 1} h_i x^i$, and $A(x) = \sum_{i \geq 1} a_i x^i$ that satisfy the following:*

*1. $C(x)$, $H(x)$, and $A(x)$ all have zero radius of convergence;*
*2. $c_n, a_n$, and $n! h_n$ are positive integers;*
*3. $A(x) = \exp(H(x)) = \exp\left(\sum_{j \geq 1} C(x^j)/j\right)$;*
*4. $\liminf_{n \to \infty} c_n / a_n = \liminf_{n \to \infty} h_n / a_n = l$.*

**Proof**. We recursively define sequences $\{N_n\}$, and $\{c_n\}$ as follows. We define $N_1 = 0$, and $c_1 = 1$. For $n > 1$, we define $N_n = [x^n] \prod_{j=1}^{n-1} (1 - x^j)^{-c_j}$ and

$$c_n = \begin{cases} n! N_n & \text{if } n \text{ is even} \\ [\frac{N_n}{\alpha - 1}] + 1 & \text{if } n \text{ is odd,} \end{cases}$$

where $\alpha = 1/l$. Notice $N_n$ and $c_n$ are positive integers for all $n > 1$. Notice that if $n$ is even, then $c_n \geq n!$ and so $C(x)$ has zero radius of convergence.

Since

$$
\begin{aligned}
[x^n]\prod_{j=1}^{\infty}(1-x^j)^{-c_j} &= [x^n](1+c_nx^n)\prod_{j=1}^{n-1}(1-x^j)^{-c_j} \\
&= c_n + N_n \\
&= c_n(1+N_n/c_n),
\end{aligned}
$$

we have that

$$
1+\sum_{j=1}^{\infty}(1+N_j/c_j)c_jx^j = \prod_{j=1}^{\infty}(1-x^j)^{-c_j}
$$

and so

$$
1+\sum_{j=1}^{\infty}(1+N_j/c_j)c_jx^j = \exp\Big(\sum_{k=1}^{\infty}C(x^k)/k\Big).
$$

Hence $a_n = (1+N_n/c_n)c_n$. Notice that

$$
\begin{aligned}
N_n &= [x^n]\prod_{j=1}^{n-1}(1-x^j)^{-c_j} \\
&\geq [x^n]\prod_{j=1}^{n-1}(1-x^j)^{-1} \\
&\geq p(n-1).
\end{aligned}
$$

Hence $N_n$ tends to infinity as $n$ tends to infinity, and so for odd $n$ we have

$$
a_n/c_n = 1+\frac{N_n}{[N_n/(\alpha-1)]+1} \to \alpha
$$

as $n$ tends to infinity. Moreover, we have that for $n$ even, $a_n/c_n = 1+1/n! \to 1$ as $n\to\infty$. Thus $C(x)\in\mathbb{Z}[[x]]$ is a power series satisfying the conditions of the theorem.

Since $H(x)=\sum_{j=1}^{\infty}C(x^j)/j$, we have that $h_n=\sum_{d|n}c_{n/d}/d$. Clearly $n!h_n$ is a positive integer for all $n\geq 1$. To complete the proof of the theorem, it suffices to show that $\lim_{n\to\infty}h_n/c_n = 1$. To see this, notice that if $n>2$, then

$$
N_n = [x^n]\prod_{j=1}^{\infty}(1-x^j)^{-c_j} \geq (1+x)^{c_1}(1+x^{n-1})^{c_{n-1}} = c_{n-1}c_1.
$$

Since $c_1 = 1$, $N_n \geq c_{n-1}$ for all $n > 1$. Thus $c_n \geq n!c_{n-1}$ for even $n$ and $c_n \geq c_{n-1}/(\alpha - 1)$ for odd $n$. It follows that $c_n \geq (n-1)!c_{n-2}/(\alpha - 1)$ for all $n > 2$, and so there is a $B > 0$ such that $c_n \geq B(n-1)!c_k$ for all $k \leq n/2$. Hence we have that for $n > 2$

$$\begin{aligned} h_n &= c_n + \sum_{\substack{d|n \\ d\neq 1}} c_{n/d}/d \\ &\leq c_n(1 + \sum_{\substack{d|n \\ d\neq 1}} 1/B(n-1)!) \\ &= c_n(1 + o(1)). \end{aligned}$$

This completes the proof of the theorem. ∎

# References


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