**A proof of a partition conjecture of Bateman and Erdős**

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Proposed Running Head: Bateman-Erdős conjecture

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**Abstract**

Bateman and Erdős found necessary and sufficient conditions on a set $A$ for the $k$'th differences of the partitions of $n$ with parts in $A$, $p_A^{(k)}(n)$, to eventually be positive; moreover, they showed that when these conditions occur $p_A^{(k+1)}(n)/p_A^{(k)}(n)$ tends to zero as $n$ tends to infinity. Bateman and Erdős conjectured that the ratio $p_A^{(k+1)}(n)/p_A^{(k)}(n) = O(n^{-1/2})$. We prove this conjecture.

Key words: Asymptotic Enumeration, partitions.

# 1 Introduction

Let $A$ be a non-empty set of positive integers. Throughout this paper, $p_A(n)$ will denote the number of partitions of $n$ with parts from $A$ and $p_A^{(k)}(n)$ will denote the $k$'th difference of $p_A(n)$. That is to say

$$p_A^{(k)}(n) = [x^n](1-x)^k \prod_{a \in A} (1-x^a)^{-1}.$$

We shall say, as do Bateman and Erdős, that a subset $A$ of the natural numbers has property $P_k$, if there are more than $k$ elements in $A$, and if any subset of $k$ elements is removed from $A$, the remaining elements have gcd one. Bateman and Erdős say that if $k < 0$, then any non-empty set of positive numbers has property $P_k$. Bateman and Erdős [1] showed that $p_A^{(k)}(n)$ is eventually positive if and only if $A$ has property $P_k$. At the end of their paper, Bateman and Erdős made a conjecture about the behavior of the $k$'th differences of $p_A(n)$. We now state this conjecture.

**Conjecture 1** *(Bateman-Erdős) If a set $A$ of positive integers has property $P_k$, then $p_A^{(k+1)}(n)/p_A^{(k)}(n) = O(n^{-1/2})$.*

Bateman and Erdős prove (see Theorem 3 of [1]) the conjecture when $A$ is a finite set; in fact, when $A$ is a finite set with property $P_k$ they show that

$$p_A^{(k+1)}(n)/p_A^{(k)}(n) = O(1/n).$$

We shall show their conjecture is true when $A$ is infinite. The best known conditions under which the Bateman-Erdős conjecture has been verified are due to Richmond [3], who, using the saddle point method, obtained asymptotics of certain partition functions that allowed him to prove the Bateman-Erdős conjecture for certain sets $A$. Bateman and Erdős observed (see page 12 of [1]) that if the conjecture were true, then it would be the best possible result. To see this, take $A = \{1, 2, 3, \ldots\}$. By Rademacher's exact formula for $p_A(n)$ (see [2]) it follows that

$$p_A^{(k+1)}(n)/p_A^{(k)}(n) = \pi(6n)^{-1/2}(1 + o(1))$$

for all $k$.

Before we begin the proof we introduce some notation.

**Notation 1** *Given a set $A$ of positive integers, we define $\pi_A(x)$ to be the number of elements of $A$ that are less than or equal to $x$.*

# 2 Acknowledgments

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# 3 Proofs

We require the following straightforward lemma.

**Lemma 1** *Suppose we have three power series*

$$A(x) = \sum_i \alpha_i x^i, \ B(x) = \sum_i \beta_i x^i, \ \text{and } C(x) = \sum_i \gamma_i x^i,$$

*where* $B(x) = (1-x^d)^{-1}A(x)$ *and* $C(x) = (1-x)^{-1}B(x)$. *Suppose also* $\alpha_n, \beta_n, \gamma_n$ *are eventually positive and that* $\alpha_n = O(\gamma_n/n)$. *Then* $\beta_n = O(\gamma_n n^{-1/2})$.

**Proof.** Take $N_0$ such that $\alpha_n, \beta_n, \gamma_n > 0$ for all $n \geq N_0$. Notice that

$$\gamma_n = \sum_{i \leq n} \beta_i$$

and hence

$$\gamma_j \geq \gamma_k \geq 0 \text{ whenever } j \geq k \geq N_0 \tag{3.1}$$

Choose a $C$ such that $\alpha_n < C\gamma_n/n$ for all $n$ greater than $N_0$. Suppose $\beta_n \neq O(\gamma_n n^{-1/2})$. Then there exists some $m$ such that $\beta_m > (C+2)\gamma_m m^{-1/2}$, $m - d\lfloor m^{1/2} \rfloor - 1 > N_0$, and

$$\lfloor m^{1/2} \rfloor (\lfloor m^{1/2} \rfloor + 1)/2(m - d\lfloor m^{1/2} \rfloor) < 1. \tag{3.2}$$

Notice

$$\alpha_m = \beta_m - \beta_{m-d} < C\gamma_m/m.$$

Hence $\beta_{m-d} > (C+2)\gamma_m m^{-1/2} - C\gamma_m/m$. Similarly,

$$\alpha_{m-d} = \beta_{m-d} - \beta_{m-2d} < C\gamma_{m-d}/(m-d).$$

Thus $\beta_{m-2d} > (C+2)\gamma_m m^{-1/2} - C\gamma_m/m - C\gamma_{m-d}/(m-d)$. And by induction we have that if $rd < m - N_0$,

$$\begin{aligned}
\beta_{m-rd} &\geq (C+2)\gamma_m m^{-1/2} - \sum_{j=0}^{r-1} C\gamma_{m-jd}/(m-jd) \\
&\geq (C+2)\gamma_m m^{-1/2} - \sum_{j=0}^{r-1} C\gamma_m/(m-rd) \qquad \text{(by (3.1))} \\
&\geq (C+2)\gamma_m m^{-1/2} - Cr\gamma_m/(m-rd). \qquad (3.3)
\end{aligned}$$

Notice if we take $q$ to be the greatest integer less than or equal to $m^{1/2}$, then we have

$$\begin{aligned}
\gamma_m &= \beta_m + \beta_{m-1} + \cdots + \beta_{m-qd} + \gamma_{m-qd-1} \\
&\geq \sum_{i=0}^{q} \beta_{m-id} \\
&\geq \sum_{i=0}^{q} \Big( (C+2)\gamma_m m^{-1/2} - Ci\gamma_m/(m-id) \Big) \qquad \text{(by (3.3))} \\
&\geq \sum_{i=0}^{q} \Big( (C+2)\gamma_m m^{-1/2} - Ci\gamma_m/(m-qd) \Big) \\
&= (C+2)(q+1)\gamma_m m^{-1/2} - q(q+1)C\gamma_m/2(m-qd) \\
&\geq (C+2)\gamma_m - C\gamma_m \qquad \text{(by (3.2))} \\
&= 2\gamma_m.
\end{aligned}$$

This is a contradiction. Hence the lemma is true. ∎

We now prove a proposition that will allow us to prove the Bateman-Erdős conjecture.

**Proposition 1** *Suppose that $A$ is a set of positive integers having property $P_k$. Then $p_A^{(k)}(n) = O(p_A^{(k-1)}(n)n^{-1/2})$.*

**Proof.** When $A$ is finite this is proven in Theorem 3 of [1]. Hence it suffices to prove the proposition when $A$ is infinite. Bateman and Erdős showed (see Lemma 2 of [1]) that $A$ has property $P_k$ if and only if some finite subset of $A$ has property $P_k$. Hence we may choose $d, e \in A$ such that $d, e > 1$ and

$$A_1 := A - \{d\} \tag{3.4}$$

and

$$A_2 := A - \{d, e\} \tag{3.5}$$

have property $P_k$. Notice

$$\begin{aligned}
(1 - x^e)(\sum_{j \geq 0} p_{A_1}^{(k)}(j)x^j) &= (1 - x^e)(1 - x)^k \prod_{a \in A_1} (1 - x^a)^{-1} \\
&= (1 - x)^k \prod_{a \in A_2} (1 - x^a)^{-1} \\
&= \sum_{j \geq 0} p_{A_2}^{(k)}(j)x^j,
\end{aligned}$$

and hence

$$p_{A_2}^{(k)}(n) = p_{A_1}^{(k)}(n) - p_{A_1}^{(k)}(n - e), \tag{3.6}$$

where $p_{A_1}^{(k)}(j)$ is understood to be zero when $j < 0$. Let

$$H(x) = \sum_{i \geq 0} h_i x^i = \log\Big((1 - x)^k (1 - x^d) \prod_{a \in A} (1 - x^a)^{-1}\Big). \tag{3.7}$$

and let

$$S(x) = \sum_{j \geq 0} s_j x^j := x(1 - x^e)^{-1} H'(x). \tag{3.8}$$

Notice

$$xH'(x) = -kx/(1-x) - dx^d/(1-x^d) + \sum_{a \in A} ax^a/(1-x^a).$$

If we consider just the positive contribution to the coefficients of $xH'(x)$, we obtain the following inequality.

$$\begin{aligned} s_n &= \sum_{\substack{j \leq n \\ j \equiv n \bmod b}} jh_j \\ &\leq \sum_{j \leq n} \sum_{\{a \in A : a | j\}} a \\ &= \sum_{\{a \in A : a \leq n\}} \sum_{j \leq n/a} a \\ &= \sum_{\{a \in A : a \leq n\}} \lfloor n/a \rfloor a \\ &\leq \sum_{\{a \in A : a \leq n\}} n \\ &= n\pi_A(n) \\ &\leq n^2. \end{aligned} \tag{3.9}$$

Similarly, if we consider only the negative contribution we find

$$s_n = \sum_{\substack{j \leq n \\ j \equiv n \bmod b}} jh_j$$

$$
\begin{aligned}
&\geq -\sum_{j\leq n} k - \sum_{m\leq n/d} d \\
&\geq -kn - dn/d \\
&= -(k+1)n. \qquad (3.10)
\end{aligned}
$$

Combining (3.9) and (3.10), we see

$$
|s_n| \leq (k+1)n^2. \qquad (3.11)
$$

From Eq. (3.7) we have

$$
\sum_{n\geq 0} p_{A_1}^{(k)}(n)x^n = \exp(H(x)). \qquad (3.12)
$$

Differentiating both sides of this equation we see

$$
\begin{aligned}
&\sum_{i\geq 1} i p_{A_1}^{(k)}(i) x^{i-1} \\
&= H'(x)\exp(H(x)) \\
&= H'(x)\sum_{i\geq 0} p_{A_1}^{(k)}(i)x^i \quad \text{(using (3.12))} \\
&= S(x)\sum_{i\geq 0}(p_{A_1}^{(k)}(i) - p_{A_1}^{(k)}(i-b))x^{i-1} \quad \text{(using (3.8))} \\
&= S(x)\sum_{i\geq 0} p_{A_2}^{(k)}(i)x^{i-1} \quad \text{(by (3.6)).} \qquad (3.13)
\end{aligned}
$$

Comparing the coefficients of $x^{n-1}$ of the first and last line of (3.13) we find

$$
np_{A_1}^{(k)}(n) = \sum_{i=0}^{n} s_{n-i} p_{A_2}^{(k)}(i). \qquad (3.14)
$$

Recall that $A_2$ has property $P_k$, and hence there exists some $N > 0$ such that $p^{(k)}_{A_2}(j) > 0$ for all $j \geq N$. Using this fact along with (3.9) and (3.11), we rewrite the right hand side of (3.14) as follows.

$$\begin{aligned} & \sum_{i=N}^{n} s_{n-i} p^{(k)}_{A_2}(i) + \sum_{i<N} s_{n-i} p^{(k)}_{A_2}(i) \\ \leq\ & \sum_{i=N}^{n} (n-i)^2 p^{(k)}_{A_2}(i) + N(k+1)n^2 (\max_{i<N} |p^{(k)}_{A_2}(i)|) \\ =\ & \sum_{i=N}^{n} (n-i)^2 p^{(k)}_{A_2}(i) + O(n^2). \end{aligned} \tag{3.15}$$

Thus there exists some $C > 0$ such that

$$np^{(k)}_{A_1}(n) \leq \sum_{i=N}^{n} (n-i)^2 p^{(k)}_{A_2}(i) + Cn^2. \tag{3.16}$$

From Lemma 1 of [1], we know that there exist $C_1, C_2 > 0$ such that

$$C_1 n^2 < [x^n](1-x)^{-1}(1-x^d)^{-1}(1-x^e)^{-1} < C_2 n^2. \tag{3.17}$$

for all $n$. Hence we can say that

$$\begin{aligned} & C_1 np^{(k)}_{A_1}(n) \\ \leq\ & \sum_{i=N}^{n} C_1 (n-i)^2 p^{(k)}_{A_2}(i) + C_1 C n^2 \\ \leq\ & \sum_{i=N}^{n} \Big( [x^{n-i}](1-x)^{-1}(1-x^d)^{-1}(1-x^e)^{-1} \Big) p^{(k)}_{A_2}(i) + C_1 C n^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n}\Big([x^{n-i}](1-x)^{-1}(1-x^d)^{-1}(1-x^e)^{-1}\Big)p_{A_2}^{(k)}(i) \\
&- \sum_{i<N}\Big([x^{n-i}](1-x)^{-1}(1-x^d)^{-1}(1-x^e)^{-1}\Big)p_{A_2}^{(k)}(i) + C_1Cn^2 \\
&\le \sum_{i=0}^{n}\Big([x^{n-i}](1-x)^{-1}(1-x^d)^{-1}(1-x^e)^{-1}\Big)p_{A_2}^{(k)}(i) \\
&+ NC_2n^2(\max_{i<N}|p_{A_2}^{(k)}(i)|) + C_1Cn^2 \\
&= \sum_{i=0}^{n}\Big([x^{n-i}](1-x)^{-1}(1-x^d)^{-1}(1-x^e)^{-1}\Big)p_{A_2}^{(k)}(i) + O(n^2). \quad (3.18)
\end{aligned}
$$

Recall that

$$p_{A_2}^{(k)}(i) = [x^i](1-x)^k(1-x^d)(1-x^e)\prod_{a\in A}(1-x^a)^{-1}.$$

Hence the last line of (3.18) can be replaced by

$$[x^n](1-x)^{k-1}\prod_{a\in A}(1-x^a)^{-1} + O(n^2) = p_A^{(k-1)}(n) + O(n^2).$$

Hence there exists a $D > 0$ such that

$$C_1np_{A_1}^{(k)}(n) \le p_A^{(k-1)}(n) + Dn^2.$$

By Theorem 5.i. of [1] we have that $n^2 = o(p_A^{(k-1)}(n))$, hence

$$p_{A_1}^{(k)}(n) = O(p_A^{(k-1)}(n)/n).$$

Notice

$$\sum_{i\ge 0} p_A^{(k)}(i)x^i = (1-x^d)^{-1}\sum_{i\ge 0} p_{A_1}^{(k)}(i)x^i.$$

and

$$\sum_{i\geq 0} p_A^{(k-1)}(i)x^i = (1-x)^{-1}\sum_{i\geq 0} p_A^{(k)}(i)x^i.$$

Applying Lemma 1, taking $\alpha_n, \beta_n$ and $\gamma_n$ to be $p_{A_1}^{(k)}(n)$, $p_A^{(k)}(n)$ and $p_A^{(k-1)}(n)$ respectively, we see that

$$p_A^{(k)}(n) = O(p_A^{(k-1)}(n)n^{-1/2}).$$

This proves the proposition. ∎

We now complete the proof of the Bateman-Erdős conjecture. To do this, we will need the following simple lemma.

**Lemma 2** *Suppose* $\{f_n\}$ *is a sequence of integers that is eventually positive and that there exists a positive integer* $c$ *such that* $f_n - f_m > 0$ *whenever* $n - m > c$. *Moreover, suppose that* $f_{n-1}/f_n \to 1$. *If*

$$\sum_{i\geq 0} h_i x^i := (1-x)(1-x^d)^{-1}(\sum_{k\geq 0} f_k x^k),$$

*then* $h_n = O(f_n)$.

**Proof.** Notice

$$h_n = \sum_{0\leq j\leq n/d} f_{n-jd} - \sum_{0\leq j\leq (n-1)/d} f_{n-1-jd}.$$

Choose an integer $r$ such that $rd - 1 > c$. we will say that $f_k = 0$ for all $k < 0$. Then we have

$$h_n \;=\; f_n + f_{n-d} + \cdots + f_{n-(r-1)d} - \sum_{0\leq j\leq -r+n/d} (f_{n-1-jd} - f_{n-jd-rd})$$

$$\begin{aligned}
& - \sum_{-r+n/d<j\le(n-1)/d} f_{n-1-jd} \\
& \le f_n + f_{n-d} + \cdots + f_{n-(r-1)d} + r \max_{j\le rd-1} |f_j| \\
& = f_n + f_{n-d} + \cdots + f_{n-(r-1)d} + O(1). \qquad (3.19)
\end{aligned}$$

Similarly,

$$h_n \ \ge\ -f_{n-1} - f_{n-1-d} - \cdots - f_{n-1-d(r-1)} + O(1). \qquad (3.20)$$

The fact that $\{f_n\}$ is a sequence of integers with the property that $f_n > f_m$ whenever $n - m > 0$ shows that $f_n \to \infty$. Combining this fact with (3.19) and (3.20) and the fact that $f_{n-1}/f_n \to 1$, we see that

$$-r \le \liminf_n h_n/f_n \le \limsup_n h_n/f_n \le r.$$

Hence $h_n = O(f_n)$ as required. ∎

We are finally ready to prove the conjecture of Bateman and Erdős.

**Theorem 1** *Conjecture 1 is correct.*

**Proof.** Suppose $A$ has property $P_k$. As stated in the introduction, we only need to prove the conjecture when $A$ is infinite. Thus we assume that $A$ is infinite. Choose $d \in A$ such that

$$A_1 := A - \{d\}$$

has property $P_k$. Since $A_1$ has property $P_k$, we have that $p_{A_1}^{(k)}(n)$ is eventually positive. Moreover, by Proposition 1 we have that

$$p_{A_1}^{(k)}(n) = O(p_{A_1}^{(k-1)} n^{-1/2}). \qquad (3.21)$$

By Theorem 6 of [1], we have that there exists a $c$ such that $p_{A_1}^{(k)}(n) - p_{A_1}^{(k)}(m) > 0$ whenever $n - m > c$. Also, $p_{A_1}^{(k)}(n-1)/p_{A_1}^{(k)}(n) \to 1$ (by Theorem 5.ii. of [1]). Furthermore, we have that

$$\sum_{i\geq 0} p_A^{(k+1)}(i)x^i \;=\; (1-x)(1-x^d)^{-1}(\sum_{j\geq 0} p_{A_1}^{(k)}(j)x^j).$$

Hence by an application of the preceding lemma, taking $f_n$ and $h_n$ to be $p_{A_1}^{(k)}(n)$ and $p_A^{(k+1)}(n)$ respectively, we see that

$$p_A^{(k+1)}(n) = O(p_{A_1}^{(k)}(n)). \tag{3.22}$$

Next notice that

$$(1-x^d)(1-x)^{-1}\sum_{i\geq 0} p_A^{(k)}(i)x^i = \sum_{j\geq 0} p_{A_1}^{(k-1)}(j)x^j.$$

Therefore

$$p_{A_1}^{(k-1)}(n) = \sum_{j=0}^{d-1} p_A^{(k)}(n-j).$$

By Theorem 5.ii. of [1], we know that for a fixed integer $j$,

$$p_A^{(k)}(n-j)/p_A^{(k)}(n) \to 1$$

as $n$ tends to infinity. Hence

$$p_{A_1}^{(k-1)}(n)/p_A^{(k)}(n) \to \sum_{j=0}^{d-1} 1 = d.$$

Combining this fact with (3.21) and (3.22), we see that

$$p_A^{(k+1)}(n) = O(p_{A_1}^{(k)}(n)) = O(p_{A_1}^{(k-1)}(n)n^{-1/2}) = O(p_A^{(k)}(n)n^{-1/2}).$$

This completes the proof of the Bateman-Erdős conjecture. /qed

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