Examples in finite Gel′fand-Kirillov dimension

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Abstract

By modifying constructions of Beĭdar and Small we prove that for countably generated prime F-algebras of finite GK dimension there exists an affinization having finite GK dimension. Using this result we show: for any field there exists a prime affine algebra of GK dimension two that is neither primitive nor PI; for any countable field F there exists a prime affine F-algebra of GK dimension three that has non-nil Jacobson radical; for any countable field F there exists an affine primitive F-algebra of GK dimension at most four with center equal to a polynomial ring; for a countable field F there exists a primitive affine Jacobson F-algebra of GK dimension three that does not satisfy the Nullstellensatz.

1 Introduction

In 1981 Beĭdar [4] gave a construction of an affine, prime algebra with non-nil Jacobson radical, answering an old question of Amitsur. Beĭdar's construction was subsequently modified by Small (unpublished). The key idea was to show that a countably-generated algebra that is not necessarily affine could appear as the corner of an affine algebra. We briefly describe Small's construction.

Let C be a commutative domain. Given a prime, countably generated C-algebra T containing C, we construct the *affinization* of R as follows. Let $R = C\{x, y\}$ and let

$$S = \begin{pmatrix} C + Ry & R \\ Ry & R \end{pmatrix}.$$

The ring S is generated as a C-algebra by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \text{ where } a \in \{1, x, y\}. \tag{1.1}$$

Hence S is an affine C-algebra. $C + Ry$ is a free C-algebra on the infinitely many generators $\{x^i y \mid i \geq 0\}$. It follows that we have a surjective ring homomorphism

$$\Phi : C + Ry \ \rightarrow \ T. \tag{1.2}$$

Let

$$P \ = \ \ker(\Phi) \tag{1.3}$$

and let $e_{i,j}$ denote the matrix with a 1 in the (i, j) entry and zeros everywhere else. Notice P is a prime ideal. Observe that $Q' := S(Pe_{1,1})S$ satisfies $e_{1,1}Q'e_{1,1} = P$. Using Zorn's lemma we can choose an ideal Q in S maximal with respect to the property that

$$e_{1,1}Qe_{1,1} \ = \ \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.4}$$

By maximality, we have that Q is prime. Observe that Q is in fact uniquely determined. To see this, suppose that Q' is another such ideal. Then

$$e_{1,1}(Q + Q')e_{1,1} \ = \ e_{1,1}Qe_{1,1} + e_{1,1}Q'e_{1,1} \ = \ P$$

By maximality of Q and Q' we have that $Q = Q + Q' = Q'$ and so $Q = Q'$. We note that

$$Q \ \supseteq \ \begin{pmatrix} 0 & 0 \\ Ry & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix} \ = \ \begin{pmatrix} 0 & 0 \\ 0 & RyPR \end{pmatrix}. \tag{1.5}$$

Similarly,

$$Q \ \supseteq \ PRe_{1,2}, \tag{1.6}$$

and

$$Q \ \supseteq \ RyPe_{2,1}. \tag{1.7}$$

The algebra S/Q has the property that

$$\overline{e_{1,1}}(S/Q)\overline{e_{1,1}} \ \cong \ T. \tag{1.8}$$

We call S/Q the *affinization of T with respect to* Φ and denote it by $\mathcal{A}(T, C; \Phi)$. We note that since the prime ideal Q is uniquely determined by Φ, the algebra $\mathcal{A}(T, C; \Phi)$ is uniquely determined by T, C, and Φ.

Given a field F and an affine algebra A, the *GK dimension* of A is defined to be

$$\text{GKdim}(A) \ := \ \limsup_{n\rightarrow\infty} \ \log\left(\dim V^n\right)/\log n,$$

where V is a finite dimensional F-subspace of A which contains the identity of A and which generates A as an F-algebra. We remark that the GK dimension is independent of choice of V. In the case that A is not affine, the GK dimension of A is defined to be the supremum of the GK dimensions of affine subalgebras of A. We refer the reader to [7] for basic facts about GK dimension. We now state the main theorem of this Chapter.

Theorem 1.1 *Let C be a commutative, affine F-domain and let T be a prime, countably generated C-algebra of GK dimension $\alpha < \infty$. Then there exists a homomorphism $\Phi : C + C\{x, y\}y \to T$ such that $\mathcal{A}(T, C; \Phi)$ has GK dimension between $\alpha + 2$ and $\alpha + 2 + \mathrm{Kdim}(C)$. In the case that $C = F$, $\mathcal{A}(T, F; \Phi)$ has GK dimension precisely $\alpha + 2$.*

Using this theorem we are able to give the following examples.

- An affine prime algebra of GK dimension 2 that is neither primitive nor PI.
- An affine prime algebra of GK dimension 3 whose Jacobson radical is not nil.
- A primitive affine algebra of GK dimension at most 4 whose center is not a field.
- A primitive affine algebra of GK dimension 3 that does not satisfy the Nullstellensatz.

2 Proofs

We first prove a lemma that will be necessary to obtain the upper bounds for our GK dimension estimates.

Lemma 2.1 *Let T be a countably generated C-algebra containing C, where C is a commutative domain and let Φ, P, and Q be as in equations (1.2), (1.3), and (1.4). Suppose $\Phi : C + C\{x, y\}y \to T$ has the property that $\Phi(x^i y) = 0$ if $i \notin \mathcal{M}$ for some set $\mathcal{M} = \{m_1, m_2, \ldots\} \subseteq \mathbb{N}$ with the property that $m_{i+1} \geq 3m_i$; also, suppose that for any m, the set*

$$\{\Phi(x^i y) \mid i \geq m\}$$

spans T as a C-module. Then

$$e_{2,2}Qe_{2,2} \;=\; RyPRe_{2,2},$$

where $R = C\{x, y\}$.

Proof. Suppose that

$$ue_{2,2} \in e_{2,2}Qe_{2,2}, \qquad u \in R. \tag{2.9}$$

We have

$$u \;=\; p(x) + \sum_{i,j\geq 0} x^i w_{i,j} x^j,$$

where $p(x) \in C[x]$ is a polynomial of degree, say d, and $w_{i,j} \in yR \cap Ry$ and at most finitely many of the $w_{i,j}$ are nonzero. Notice that

$$(x^k e_{1,2})(ue_{2,2})(x^\ell y e_{2,1}) \;=\; x^k u x^\ell y e_{1,1} \;\in\; e_{1,1}Qe_{1,1} \;=\; Pe_{1,1}. \tag{2.10}$$

Hence

$$\Phi(x^{k+\ell}p(x)y) + \sum_{i,j} \Phi(x^{k+i} w_{i,j} x^{j+\ell} y) \;=\; 0.$$

Since $m_{i+1} - m_i \to \infty$ and $w_{i,j}$ is zero for all but finitely many pairs (i, j), we have that for (i_0, j_0) there exists some index N such that for any $\ell_1, \ell_2 \geq N$ we have

$$\Phi(x^{m_{\ell_1} - i_0 + i} y) \;=\; 0 \qquad \text{for } i \neq i_0, \text{ and} \tag{2.11}$$

$$\Phi(x^{m_{\ell_2} - j_0 + j} y) \;=\; 0 \qquad \text{for } j \neq j_0. \tag{2.12}$$

Using these equations and the fact that $w_{i,j} \in yR \cap Ry$, we see that for any $\ell_1, \ell_2 \geq N$, $(x^{m_{\ell_1} - i_0 + i}) w_{i,j} (x^{m_{\ell_2} - j_0 + j} y) \in P$ whenever $(i, j) \neq (i_0, j_0)$. Hence

$$x^{m_{\ell_1} + m_{\ell_2} - i_0 - j_0} p(x) y + x^{m_{\ell_1}} w_{i_0,j_0} x^{m_{\ell_2}} y \in P,$$

whenever $\ell_1, \ell_2 \geq N$. It follows that if ℓ_1 and ℓ_2 are sufficiently large, then

$$\begin{aligned} m_{\max(\ell_1,\ell_2)} &< m_{\ell_1} + m_{\ell_2} - i_0 - j_0 \\ &\leq m_{\ell_1} + m_{\ell_2} - i_0 - j_0 + d \\ &\leq 2m_{\max(\ell_1,\ell_2)} + d \\ &< 3m_{\max(\ell_1,\ell_2)} \leq m_{\max(\ell_1,\ell_2)+1}. \end{aligned}$$

Hence

$$\Phi(x^{m_{\ell_1}+m_{\ell_2}-i_0-j_0}p(x)y) \;=\; 0$$

for all ℓ_1, ℓ_2 sufficiently large. Thus we have that

$$x^{m_{\ell_1}} w_{i_0,j_0} x^{m_{\ell_2}} y \in P$$

for all ℓ_1, ℓ_2 sufficiently large. Since $w_{i_0,j_0} \in yR \cap Ry$ we can write $w_{i_0,j_0} = yv$ with $v \in C + Ry$. Then there exists a positive integer N' such that

$$0 = \Phi(x^{m_{\ell_1}} w_{i_0,j_0} x^{m_{\ell_2}} y) \;=\; \Phi(x^{m_{\ell_1}} y)\Phi(v)\Phi(x^{m_{\ell_2}} y)$$

for all $\ell_1, \ell_2 > N'$. Since

$$\{\Phi(x^{m_{\ell_1}} y) \mid \ell \geq N'\}$$

spans T as a C-module, letting ℓ_1 and ℓ_2 range independently over all natural numbers greater than N', we find that $T\Phi(v)T = 0$ and hence $v \in P$. It follows that $x^{i_0} w_{i_0,j_0} x^{j_0} \in RyPR$ for all i_0, j_0. Hence $u \equiv p(x) \bmod RyPR$. It follows from equation (1.5) that $p(x)e_{2,2} \in Q$, and so from equation (2.10), with $p(x)$ replacing u, we have $p(x)x^i y \in P$ for all i. We claim that this implies that $p(x) = 0$. Suppose that $p(x) \neq 0$ and write $p(x) = p_0 + \cdots + p_d x^d$ with $p_d \neq 0$. Using the fact that $m_{i+1} - m_i \to \infty$, we see that for j sufficiently large $\Phi(p(x)x^{m_j - d}y) = p_d \Phi(x^{m_j} y) \neq 0$, a contradiction since p_d is a nonzero central element and is therefore regular. Hence $p(x) = 0$ and so $u \in RyPR$. Thus $e_{2,2}Qe_{2,2} \subseteq RyPRe_{2,2}$. From equation (1.5) we have that $e_{2,2}Qe_{2,2} = RyPRe_{2,2}$. ∎

Theorem 2.2 *Let T be a prime, countably generated F-algebra of GK dimension $\alpha < \infty$. Then there exists a homomorphism $\Phi : F + F\{x,y\}y \to T$ such that $\mathcal{A}(T, F; \Phi)$ has GK dimension $\alpha + 2$.*

Proof. Let P and Q be as in equations (1.3) and (1.4). Let $R = F\{x,y\}$, let $V \subseteq S$ be the vector space spanned by the generating set given in item (1.1), and let

$$W \;=\; F + Fx + Fy \;\subseteq\; R. \tag{2.13}$$

One has

$$V^n \;\subseteq\; \begin{pmatrix} F + W^n y & W^n \\ W^n y & W^n \end{pmatrix}.$$

We shall construct a homomorphism that will give an affinization of finite GK dimension. Let $\mathcal{B} = \{1, u_1, u_2 \ldots\} \subseteq T$ have the property that $\{u_i \mid i \geq m\}$ spans T as an F-vector space for any m. For example, if $\{v_1, v_2, \ldots\}$ is a basis for T as a F-vector space, we can take

$$\mathcal{B} \;=\; \{1, v_1, v_1, v_2, v_1, v_2, v_3, \ldots, v_1, v_2, \ldots, v_n, v_1, v_2, \ldots, v_{n+1}, \ldots\}.$$

For $j \geq 0$, define U_j to be the vector space spanned by the first $j+1$ elements of $\mathcal{B}$; that is,

$$U_j \;=\; F + Fu_1 + \cdots + Fu_j. \tag{2.14}$$

Let $\varepsilon > 0$. Since T has GK dimension α, we have

$$\limsup_{n\to\infty} \; \log(\dim\,(U_j)^n)\Big/\log n \;\leq\; \alpha.$$

Hence there exists a positive integer m_j such that

$$\dim\,(U_j)^n \;<\; n^{\alpha+\varepsilon}$$

for all $n \geq m_j$. By increasing m_j if necessary, we may assume that $m_j \geq 3m_{j-1}$ for all $j \geq 2$. We define

$$\Phi : F + F\{x,y\}y \;\to\; T$$

by

$$\Phi(x^i y) \;=\; \begin{cases} u_j & \text{if } i = m_j \text{ for some } j \geq 0, \\ 0 & \text{if } i \neq m_j \text{ for all } j \geq 0. \end{cases}$$

Consider

$$\dim \begin{pmatrix} F + W^n y & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose $m_j \leq n < m_{j+1}$. Then since $Pe_{1,1} = e_{1,1}Qe_{1,1}$, a word in $W^n y\overline{e_{1,1}}$ is determined by its behavior modulo $Pe_{1,1}$. Since $n < m_{j+1}$, we have

$$\Phi(W^n y) \;\subseteq\; (U_j)^n.$$

Hence

$$\dim W^n y\overline{e_{1,1}} \;\leq\; \dim\,(U_j)^n \;\leq\; n^{\alpha+\varepsilon}.$$

We now compute the dimension of $W^n\overline{e_{2,2}}$. Notice anything in the ring R can be expressed as a linear combination of powers of x, elements of the form

x^iyx^j, and elements of the form x^iywyx^j, where w is a word in x and y and $i, j \geq 0$. Hence anything in W^n is contained in the span of

$$\{1, x, x^2, \ldots, x^n\} \cup \{x^iyx^j \mid 0 \leq i, j \leq n\} \cup \{x^iyW^nyx^j \mid 0 \leq i, j \leq n\}.$$

The dimensions of $\mathrm{Span}\{1, x, x^2, \ldots, x^n\}$ and $\mathrm{Span}\{x^iyx^j \mid 0 \leq i, j \leq n\}$ are $(n+1)$ and $(n+1)^2$ respectively. Now $RyPRe_{2,2} \subseteq Q$ and hence the image in $\overline{e_{2,2}}\mathcal{A}(T, F; \Phi)\overline{e_{2,2}}$ of an element of the form $x^iywyx^je_{2,2}$ is completely determined by the behavior of wy mod P. As $\Phi(W^ny) \subseteq (U_j)^n$, we have for $n \geq m_j$,

$$\dim\, x^iyW^nyx^j\overline{e_{2,2}} \;\leq\; \dim\, (U_j)^n \;\leq\; n^{\alpha+\varepsilon}.$$

Since i, j can assume any value between 0 and n,

$$\dim\, W^n\overline{e_{2,2}} \;\leq\; (n+1) + (n+1)^2 + (n+1)^2n^{\alpha+\varepsilon} \;=\; \mathrm{O}(n^{2+\alpha+\varepsilon}).$$

Let A denote the "diagonal" of $\mathcal{A}(T, F; \Phi)$ and let B denote the "upper-triangular part" of $\mathcal{A}(T, F; \Phi)$. We have just shown that

$$A \;\cong\; ((F + Ry)/P) \oplus (R/RyPR)$$

has GK dimension at most $\alpha + 2 + \varepsilon$. Observe that $B = A + \overline{e_{1,2}}A$ and hence B has GK dimension at most $\alpha + 2 + \varepsilon$ by Lemma 4.3 of [7]. Finally, note that

$$\mathcal{A}(T, F; \Phi) \;=\; B + B(y\overline{e_{2,1}})$$

and thus

$$\mathcal{A}(T, F; \Phi)$$

has GK dimension at most $\alpha + 2 + \varepsilon$, again using Lemma 4.3 of [7]. Since $\varepsilon > 0$ is arbitrary, we conclude that $\mathcal{A}(T, F; \Phi)$ has GK dimension at most $\alpha + 2$.

On the other hand, since T has GK dimension α and $\mathcal{B}$ spans T, there exists j such that

$$\limsup_{n\to\infty}\; (\dim\, (U_j)^n)/n^{\alpha-\varepsilon} = \infty.$$

We have $V^{m_jn+3n+2} \supseteq W^{m_jn+3n+2}e_{2,2}$. Note

$$\begin{aligned} & W^{m_jn+3n+2} \\ \supseteq\;\; & \{x^kywyx^\ell \mid 0 \leq k, \ell \leq n \text{ and } w \text{ is a word of length } \leq m_jn + n\}. \end{aligned}$$

By Lemma 2.1 $e_{2,2}Qe_{2,2} = RyPR$. Since $PRy, RyP \subseteq P$, $RyPR$ is spanned by elements of the form

$$\{x^i yax^j \mid a \in P,\ i,j \geq 0\}. \tag{2.15}$$

We claim the dimension of

$$\mathrm{Span}\{x^k yW^{m_jn+n}yx^\ell \mid 0 \leq k,\ell \leq n\}\overline{e_{2,2}}$$

is just $(n+1)^2$ times the dimension of $W^{m_jn+n}y$ modulo the ideal P. To see this, note that the sum

$$\sum_{k,\ell=0}^{\infty} x^k yRyx^\ell$$

is direct, as R is free. Let $q = m_jn + n$, and set

$$E \ = \ \sum_{k,\ell=0}^{n} x^k yW^q yx^\ell.$$

Since $RyPR = \bigoplus_{k,\ell=0}^{\infty} x^k yPx^\ell$,

$$E \cap RyPR \ = \ \bigoplus_{k,\ell=0}^{n} x^k y(W^q y \cap P)x^\ell.$$

Hence

$$E\overline{e_{2,2}} \ \cong \ E/(E \cap RyPR) \ \cong \ \bigoplus_{k,\ell=0}^{n} \Big(W^k y/(W^q y \cap P)\Big),$$

which yields the claim.

Since $\Phi(W^{m_j}y) \supseteq U_j$ and $W^{m_jn+n}y \supseteq (W^{m_j}y)^n$, we have

$$\Phi(W^{m_jn+n}y) \ \supseteq \ \Phi(W^{m_j}y)^n \ \supseteq \ (U_j)^n.$$

Hence the dimension of $W^{m_jn+n}y$ mod P is at least dim $(U_j)^n$. Thus

$$\begin{aligned}
\dim \overline{V^{m_jn+3n+2}} \ &\geq \ \dim W^{m_jn+3n+2}\overline{e_{2,2}} \\
&\geq \ (n+1)^2 \dim \Big(W^{m_jn+n}y/W^{m_jn+n}y \cap P\Big) \\
&\geq \ (n+1)^2 \dim\,(U_j)^n.
\end{aligned}$$

But

$$\limsup_{n\to\infty} \ (\dim\ (U_j)^n)/n^{\alpha-\varepsilon} \ = \ \infty,$$

and so

$$\limsup_{n\to\infty} \ (\dim\ \overline{V^{m_j n+3n+2}})/(m_j n+3n+2)^{\alpha-\varepsilon} \ = \ \infty.$$

Since

$$\begin{aligned}
\mathrm{GKdim}(\mathcal{A}(T,F;\Phi)) &= \limsup_{n\to\infty} \ \log\dim(\overline{V^n})/\log(n) \\
&\geq \limsup_{n\to\infty} \log\dim(\overline{V^{m_j n+3n+2}})/\log(m_j n+3n+2) \\
&\geq \limsup_{n\to\infty} \log\left((n+1)^2 \dim\ (U_j)^n\right)/\log(m_j n+3n+2) \\
&\geq \limsup_{n\to\infty} \log\left((n+1)^2 n^{\alpha-\varepsilon}\right)/\log(m_j n+3n+2) \\
&\geq \limsup_{n\to\infty} (\alpha+2-\varepsilon)\log(n)/\log(m_j n+3n+2) \\
&\geq \alpha+2-\varepsilon.
\end{aligned}$$

Hence the GK dimension of $\mathcal{A}(T,F;\Phi) \geq 2+\alpha-\varepsilon$. It follows that $\mathcal{A}(T,F;\Phi)$ has GK dimension precisely equal to $2+\alpha$. ∎

Corollary 2.3 *Suppose T is a countably generated C-algebra containing C, where C is a commutative affine F-algebra which is a domain. Then there exists $\Phi : C + C\{x,y\}y \to T$ such that*

$$\alpha+2 \ \leq \ \mathrm{GKdim}(\mathcal{A}(T,C;\Phi)) \ \leq \ 2+\alpha+\mathrm{Kdim}(C).$$

Proof. Let $R = F\{x,y\}$, $R' = CR$,

$$S \ = \ \begin{pmatrix} F+Ry & R \\ Ry & R \end{pmatrix} \qquad \text{and} S' \ = \ \begin{pmatrix} C+R'y & R' \\ R'y & R' \end{pmatrix} \ = \ CS.$$

Given $\Phi : F + Ry \to T$ as in Theorem 1.1, we may extend Φ to $\Phi' : C + R'y \to T$ by $\Phi'(c) = c$ for all $c \in C$. Let $P' = \ker(\Phi')$ and $P = \ker(\Phi) = P' \cap (F + Ry)$. Use Zorn's lemma to choose an ideal Q' of S' maximal such that $e_{1,1}Q'e_{1,1} = P'$. Then $e_{1,1}(Q' \cap S)e_{1,1} = P' \cap (F + Ry) = P$. Choose an ideal Q of S maximal such that $Q \supseteq Q' \cap S$ and $e_{1,1}Qe_{1,1} = P$. Then $S'QS' = CQ$ is an ideal of S' and $e_{1,1}CQe_{1,1} = CP \subseteq P'$. So,

$$e_{1,1}(CQ + Q')e_{1,1} \;=\; e_{1,1}CQe_{1,1} + e_{1,1}Q'e_{1,1} \;=\; CP + P' \;\subseteq\; P'.$$

By the maximality of Q', $CQ + Q' = Q'$, so $Q \subseteq Q' \cap S \subseteq Q$, so $Q' \cap S = Q$. Thus

$$\mathcal{A}(T, F; \Phi) \;=\; S/Q \;\subseteq\; S'/Q' \;=\; \mathcal{A}(T, C; \Phi').$$

From this we see

$$\alpha + 2 \;=\; \text{GKdim}(\mathcal{A}(T, F; \Phi)) \;\leq\; \text{GKdim}(\mathcal{A}(T, C; \Phi')).$$

Let $\{1, c_1, \ldots, c_m\}$ be a generating set for C as an F-algebra. Let V be the image in $\mathcal{A}(T, F; \Phi)$ of the vector space spanned by the generating set given in item (1.1), then

$$\limsup_{n\to\infty} \; \log(\dim V^n)/\log n \;=\; \alpha + 2. \tag{2.16}$$

Let $\mathbf{I_2} \in \mathcal{A}(T, C; \Phi')$ denote the identity matrix. We have

$$Y \;:=\; \sum_{i=1}^{m} Fc_i\mathbf{I_2} + V$$

generates $\mathcal{A}(T, C, \Phi')$ as an F-algebra. Notice

$$V^n \;\subseteq\; Y^n \;\subseteq\; \Big(\sum_{i=1}^{m} Fc_i\mathbf{I_2}\Big)^n V^n. \tag{2.17}$$

Also,

$$\dim \Big(\sum_{i=1}^{m} Fc_i\mathbf{I_2}\Big)^n \; V^n \tag{2.18}$$

$$\leq \;\; \bigg(\dim\Big(\sum_{i=1}^{m} Fc_i\mathbf{I_2}\Big)^n\bigg)\Big(\dim V^n\Big).$$

Since the GK dimension of C is the same as the Krull dimension of C (see Theorem 4.5 on page 40 of [7]), we have that for any $\varepsilon > 0$

$$\dim \Big(\sum_{i=1}^{m} Fc_i\mathbf{I_2}\Big)^n \;\leq\; n^{\text{Kdim}(C)+\varepsilon}$$

for all n sufficiently large. Combining this fact with equations (2.16), (2.17), and (2.18) we see that $\mathcal{A}(T, C; \Phi')$ has GK dimension at most $\alpha+2+K\dim C$. ∎

We now compute the center of $\mathcal{A}(T, C; \Phi)$.

Proposition 2.4 *Let Φ be as in Corollary 2.3. Then the center of $\mathcal{A}(T, C; \Phi)$ is $\{c\mathbf{I_2} \mid c \in C\}$.*

Proof. Let $R = C\{x, y\}$ and let

$$\mathbf{z} \;=\; \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{pmatrix}$$

be a central element of $\mathcal{A}(T, C; \Phi)$. Then

$$z_{2,1} \;=\; \overline{e_{2,2}}\mathbf{z}\overline{e_{1,1}} \;=\; \mathbf{z}\overline{e_{2,2}e_{1,1}} \;=\; 0.$$

Similarly, $z_{1,2} = 0$. Thus

$$\mathbf{z} \;=\; \begin{pmatrix} z_{1,1} & 0 \\ 0 & z_{2,2} \end{pmatrix}.$$

Write

$$\tilde{z}_{2,2} \;=\; p(x) + \sum_{i,j\geq 0} x^i w_{i,j} x^j \tag{2.19}$$

such that $\tilde{z}_{2,2}$ maps to $z_{2,2}$ in $RyPR$ and with $p(x) \in C[x]$ and $w_{i,j} \in Ry \cap yR$ and only finitely many of the $w_{i,j}$ nonzero. We have

$$z_{2,2}x^m\overline{e_{2,2}} \;=\; \mathbf{z}(x^m\overline{e_{2,2}}) \;=\; (x^m\overline{e_{2,2}})\mathbf{z} \;=\; x^m z_{2,2}\overline{e_{2,2}}.$$

By Lemma 2.1, $\tilde{z}_{2,2}x^m - x^m\tilde{z}_{2,2} \in RyPR$ and so

$$\tilde{z}_{2,2}x^m x^n y - x^m \tilde{z}_{2,2} x^n y \in P \tag{2.20}$$

for all $m, n \geq 0$. By assumption, there exists a set $\mathcal{M} = \{m_1, m_2, \ldots\}$ with $m_{i+1} - m_i \to \infty$ and $\Phi(x^i y) = 0$ if $i \notin \mathcal{M}$. Using equations (2.19) and (2.20), we argue as in the proof of Lemma 2.1, fixing i_0 and j_0 and taking $n = m_{\ell_2} - j_0$ and $n' = m_{\ell_1} - i_0$ for some large ℓ_1 and ℓ_2. If ℓ_1 and ℓ_2 are sufficiently large we have

$$\Phi(x^{n'}\tilde{z}_{2,2}x^n y) \;=\; \Phi(x^{m_{\ell_1}} w_{i_0,j_0} x^{m_{\ell_2}} y). \tag{2.21}$$

On the other hand,

$$\tilde{z}_{2,2}x^{n'}x^n y \;=\; \tilde{z}_{2,2}x^{m_{\ell_1}+m_{\ell_2}-i_0-j_0}y.$$

Choose N_1 such that $w_{i,j} = 0$ for $i, j \geq N_1$. Then if ℓ_1, ℓ_2 are sufficiently large,

$$m_{\max(\ell_1,\ell_2)} \;<\; m_{\ell_1}+m_{\ell_2}-i_0-j_0 \;\leq\; m_{\ell_1}+m_{\ell_2}-i_0-j_0+N' \;<\; m_{\max(\ell_1,\ell_2)+1}$$

and hence

$$\Phi(\tilde{z}_{2,2}x^{n'}x^n y) \;=\; 0. \tag{2.22}$$

Thus using equations (2.20), (2.21) and (2.22), we see

$$\tilde{z}_{2,2}x^{n'}x^n y - x^{n'}\tilde{z}_{2,2}x^n y \;\equiv\; x^{m_{\ell_2}}w_{i_0,j_0}x^{m_{\ell_1}}y \mod P.$$

Since $w_{i_0,j_0} \in Ry \cap yR$, there exists $v \in C + Ry$ such that $w_{i_0,j_0} = yv$. It follows that there exists a positive integer N_2 such that

$$0 \;=\; \Phi(x^{m_{\ell_2}}y)\Phi(v)\Phi(x^{m_{\ell_1}}y)$$

for all $\ell_1, \ell_2 > N_2$. By letting ℓ_1 and ℓ_2 range independently over all natural numbers greater than N' we see that $R\Phi(v)R = 0$. Hence $v \in P$. It follows that $x^{i_0}w_{i_0,j_0}x^{j_0} \in RyPR$. Thus $\tilde{z}_{2,2} \equiv p(x) \bmod RyPR$. We may therefore assume that $\tilde{z}_{2,2} = p(x)$ by item (1.5). Write $p(x) = p_0 + \cdots + p_d x^d$. We have

$$x^n y p(x) x^{n'} y - p(x) x^n y x^{n'} y \in P \qquad \text{for all } n, n' \geq 0. \tag{2.23}$$

Fix $i > 0$. We take $n = m_{\ell_1} - i$ and $n' = m_{\ell_2}$. Since $m_{i+1} - m_i \to \infty$, we have

$$\Phi(x^{m_\ell - j}y) \;=\; \Phi(x^{m_\ell - i + j}y) \;=\; 0$$

for $0 \leq j \leq d$, $j \neq i$, for all ℓ sufficiently large. Hence

$$0 \;=\; \Phi(x^{m_{\ell_1}-i}yp(x)x^{m_{\ell_2}}y - p(x)x^{m_{\ell_1}-i}yx^{m_{\ell_2}}y) \;=\; -p_i\Phi(x^{m_{\ell_1}}y)\Phi(x^{m_{\ell_2}}y).$$

By allowing ℓ_1 and ℓ_2 to range independently over all sufficiently large numbers, we see that $p_i = 0$ for all $i > 0$. Hence $z_{2,2} = c \in C$. Now $c\mathbf{I_2}$ is central and hence

$$\mathbf{z} - c\mathbf{I_2} \;=\; (z_{1,1} - c)\overline{e_{1,1}}$$

is central. But the nonzero central elements of a prime ring are regular and $(z_{1,1}-c)\overline{e_{1,1}}$ is annihilated by $\overline{e_{2,2}}$. We conclude that $z_{1,1}=c$ and so $\mathbf{z}=c\mathbf{I_2}$. ∎

An important fact about the affinization of a C-algebra T is that T is primitive if and only if $\mathcal{A}(T,C;\Phi)$ is primitive. In fact, $T \cong e_{1,1}\mathcal{A}(T,C;\Phi)e_{1,1}$, and so this fact is just a consequence of the following proposition.

Proposition 2.5 *(Lanski, Resco, Small [8]) Let R be a prime ring with a nonzero idempotent e. Then R is primitive if and only if eRe is primitive.*

Proof. Suppose R is primitive. Let M be a faithful simple R-module. Notice that eM is an eRe-module. Note that if $ere \in eRe$ annihilates eM, then $0=(ere)eM=(ere)M$ and so $ere=0$. Hence eM is a faithful eRe-module. Suppose $ev \in eM$ is nonzero. Then since $ev \in M$ and M is simple as an R-module, we have $Rev=M$. Thus $(eRe)ev=eM$ and so we see that eM is a simple eRe-module. Hence eRe is primitive.

Suppose eRe is primitive. Then there is a maximal left ideal $\mathcal{M} \subseteq eRe$ that does not contain a nonzero two-sided eRe-ideal. Notice $I := R\mathcal{M} + R(1-e)$ has the property that $eIe=\mathcal{M}$. By Zorn's lemma we can find a left R-ideal $\mathcal{M}'$ that contains I and is maximal with respect to the property that $e\mathcal{M}'e=\mathcal{M}$. The ideal $\mathcal{M}'$ is necessarily a maximal left-ideal, because if $\mathcal{M}'+Rx$ properly contains $\mathcal{M}'$, then $eRe=\mathcal{M}+eRxe$; hence

$$e \;=\; v+erxe \tag{2.24}$$

for some $v \in \mathcal{M}$ and some $r \in R$. Since $v \in \mathcal{M} \subset eRe$, we have that $v=ve$ and so equation (2.24) gives that $(1-v-erx)e=0$. Thus $1-v-erx \in R(1-e)$. It follows that

$$1 \in R\mathcal{M}+R(1-e)+Rx \;\subseteq\; \mathcal{M}'+Rx$$

and so $\mathcal{M}'$ is a maximal left ideal. Suppose $\mathcal{M}'$ contains a nonzero two-sided ideal J. Then eJe is a two-sided eRe-ideal contained in $\mathcal{M}$. Hence $eJe=0$. But this is impossible because R is prime, J is nonzero, and e is nonzero. Thus $\mathcal{M}'$ cannot contain a nonzero two-sided ideal. Hence R is primitive. ∎

3 Applications

We now give some applications of Theorem 1.1.

Example 3.1 *An affine prime F-algebra of GK dimension three with non-nil Jacobson radical for countable fields F.*

Suppose F is a countable field. Beĭdar [4] first constructed an affine prime ring with non-nil Jacobson Radical; this example was subsequently modified by Small. We show that Small's construction can be further modified to give such an example with GK dimension three. Note that $F[t]_{(t)}$, the localization of $F[t]$ at the prime ideal (t), is countably infinite dimensional over F and has GK dimension one as an F-algebra. Hence by Theorem 1.1 there is a homomorphism $\Phi : F + F\{x, y\}y \to F[t]_{(t)}$ such that $A = \mathcal{A}(F[t]_{(t)}, F; \Phi)$ is an affine algebra with GK dimension 3. We have that

$$\overline{e_{1,1}}J(A)\overline{e_{1,1}} \supseteq J(\overline{e_{1,1}}A\overline{e_{1,1}}) \cong J(F[t]_{(t)}) = tF[t]_{(t)}.$$

Hence A has non-nil Jacobson radical. This is much different from the situation when F is uncountable. In this case a trick of Amitsur (see [1]) shows that the Jacobson radical of any affine F-algebra is nil. Also, a prime affine ring of GK dimension less than or equal to one is PI by a theorem of Small and Warfield [10] and hence has Jacobson radical (0). It is unknown whether the Jacobson radical of a prime, affine ring of GK dimension two is necessarily nil. We mention the following conjecture of Goodearl.

Conjecture 3.1 *Suppose R is an affine F-algebra of GK dimension 2. Then $J(R)$ is nil. In particular, if R is also right Goldie then R is Jacobson.*

The affinization technique will probably not yield a counter-example to this conjecture, because it will only produce a ring of GK dimension less than three if the affinization is performed upon a ring of GK dimension zero; if R is a locally finite F-algebra then R is algebraic over F and hence $J(R)$ is nil.

Example 3.2 *A primitive F-algebra with GK dimension at most 4 and center equal to a polynomial ring for countable fields F.*

Let F be a countable field. Consider the field $F(t)$. This is a primitive countably generated $F[t]$-algebra. By Theorem 1.1 there exists a homomorphism

$\Phi : F[t] + F[t]\{x, y\} \to F(t)$ such that $\mathcal{A}(F(t), F[t]; \Phi)$ has GK dimension at most four. By Proposition 2.4 we see that the center of this ring is the polynomial ring $F[t]\mathbf{I_2}$. Thus $\mathcal{A}(F(t), F[t]; \Phi)$ is a primitive affine F-algebra of GK dimension at most four with center equal to a polynomial ring.

Example 3.3 *An affine ring of GK dimension two that is neither PI nor primitive.*

Let F be any field. We shall construct our example by first finding a locally finite, prime F-algebra that is not primitive, and then we shall use affinization on this example. Let us create a free F-algebra on infinitely many generators $\{t_1, t_2, \dots\}$. We let W_n be the collection of all words of length n on $t_1, \dots, t_n$ and let

$$I \;=\; \left\langle \bigcup_{i=1}^{\infty} W_i \right\rangle.$$

Remark 3.1 *Note that I is a monomial ideal and hence if $c_1, \dots, c_\ell$ are nonzero and $w_1, \dots, w_\ell$ are distinct words in $t_1, t_2, \dots$, then*

$$\sum_{i=1}^{\ell} c_i w_i \in I$$

only if $w_1, \dots, w_\ell \in I$.

Let

$$R \;=\; F\{t_1, t_2, \dots\}/I.$$

We prove two lemmas to establish the facts that we shall need.

Lemma 3.2 *R is not PI and is locally finite.*

Proof. To see that R is not PI, suppose that R satisfies a multilinear identity

$$p(x_1, x_2, \dots, x_n) \;=\; x_1 x_2 \cdots x_n + \sum_{\substack{\sigma \in S_n \\ \sigma \neq 1}} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Consider $p(t_2, t_3, \dots, t_n)$. By Remark 3.1 we have that $t_2 t_3 \cdots t_{n+1} \in I$. Since $t_2 t_3 \cdots t_{n+1}$ is a word in I, we have that there is a word $w \in W_i$ for some $i \leq n$ and words v and v' such that

$$t_2 \cdots t_{n+1} \;=\; v w v'.$$

Thus $w = t_i t_{i+1} \cdots t_{i+r}$ for some i and r with $2 \leq i \leq i + r \leq n + 1$. Since w has length $r + 1$, we conclude that $w \in W_{r+1}$. This is a contradiction, since t_{i+r} appears in w and $i + r \geq r + 2$. Hence R does not satisfy a polynomial identity.

To see that R is locally finite note that if $p_1, \ldots, p_m \in F\{t_1, t_2, \ldots\}$, then there exists some k such that

$$p_1, \ldots, p_m \in F\{t_1, \ldots, t_k\} \subseteq F\{t_1, t_2, \ldots\}.$$

Since

$$I \cap F\{t_1, \ldots, t_k\}$$

contains all words in $t_1, \ldots, t_k$ of length at least k, we conclude that

$$F\{t_1, \ldots, t_k\}/(I \cap F\{t_1, \ldots, t_k\})$$

has dimension at most $1 + k + k^2 + \cdots + k^{k-1}$. Hence R is locally finite. ∎

Lemma 3.3 *R is prime and $J(R) = < t_1, t_2, \ldots >$. In particular, R is not primitive.*

Proof. Suppose $a, b \in F\{t_1, t_2, \ldots\}$ are such that

$$a\Big(F\{t_1, t_2, \ldots\}\Big)b \subseteq I$$

and $a, b \notin I$. It is no loss of generality to assume that a and b are words, by Remark 3.1. We then have $at_m b \in I$ for all m. Let length(u) denote the length of a word u and choose

$$m > 1 + \text{length}(a) + \text{length}(b).$$

Since $at_m b$ is a word that is in I, there exists

$$w \in \bigcup_{i \geq 1} W_i$$

and words v, v' such that $at_m b = vwv'$. If t_m does not occur in w, then w is a subword of either a or b, contradicting the fact that $a, b \notin I$. Hence t_m appears in the word w. It follows that

$$\text{length}(w) \geq m > 1 + \text{length}(a) + \text{length}(b) = \text{length}(at_m b).$$

This is a contradiction, and so we see that R is prime. Let

$$p \in < t_1, t_2, \dots > \subseteq F\{t_1, t_2, \dots\}.$$

Then there exists a k such that

$$p \in F\{t_1, \dots t_k\} \subseteq F\{t_1, t_2, \dots\}.$$

We have that p^k is a linear combination of words in $t_1, \dots, t_k$, all of which have length at least k. Hence $p^k \in I$. It follows that the Jacobson radical of R is the ideal generated by the images of $\{t_i : i \geq 1\}$. ∎

From these lemmas we see that R is a prime ring of GK dimension zero and is neither PI nor primitive; moreover, R is countable dimensional as an F-algebra. Hence by Theorem 1.1 there exists a homomorphism $\Phi : F + F\{x, y\} \to R$ such that $\mathcal{A}(R, F; \Phi)$ is a prime ring of GK dimension two. Since R is not primitive, $\mathcal{A}(R, F; \Phi)$ is not primitive by Proposition 2.5. Also $\mathcal{A}(R, F; \Phi)$ is not PI, as R does not satisfy a polynomial identity.

Fisher and Snider (see Example 2.8 of [5]) give an example of a prime locally finite F-algebra that has nonzero Jacobson radical. It can be shown that their example is not PI and hence we obtain via affinization another example of a prime affine ring of GK dimension two that is neither primitive nor PI. In both the example given here and the example that arises from using the Fisher and Snider example we have that the degrees of the matrix images are bounded. We therefore repeat the following questions of Small.

Question 3.1 *Are the degrees of the matrix images of a prime, affine ring of GK dimension two necessarily bounded?*

Question 3.2 *Must an affine algebra with quadratic growth necessarily be either primitive or PI?*

Example 3.4 *A primitive affine F-algebra of GK dimension three that does not satisfy the Nullstellensatz for countable fields F.*

We recall that an F-algebra A satisfies the *Nullstellensatz* if the Jacobson radical of every factor ring of A is nil and for every simply A-module M, $\mathrm{End}_A(M)$ is algebraic over F. Suppose F is a countable field. Notice $F(t)$ is a countably generated primitive F-algebra of GK dimension 1. Hence by

Theorem 1.1 and Proposition 2.5 there exists $\Phi : F + F\{x,y\}y \to F(t)$ such that $\mathcal{A}(F(t),F;\Phi)$ is a primitive affine F-algebra of GK dimension three. We choose a maximal left ideal $\mathcal{M}$ of $\mathcal{A}(F(t),F;\Phi)$ that contains the image of the left ideal

$$\begin{pmatrix} 0 & F\{x,y\} \\ 0 & F\{x,y\} \end{pmatrix}.$$

Suppose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}.$$

Then left multiplication by $\overline{e_{1,1}}$ gives that $a\overline{e_{1,1}} + b\overline{e_{1,2}} \in \mathcal{M}$. Since $b\overline{e_{1,2}} \in \mathcal{M}$, we see that $a\overline{e_{1,1}} \in \mathcal{M}$. But

$$\overline{e_{1,1}}\mathcal{A}(F(t),F[t];\Phi)\overline{e_{1,1}} \;\cong\; F(t)$$

is simple and hence if $a \neq 0$, we must have $\overline{e_{1,1}} \in \mathcal{M}$ and so $\mathbf{I_2} = \overline{e_{1,1}} + \overline{e_{2,2}} \in \mathcal{M}$. Thus we conclude that $a = 0$ and so

$$\mathcal{A}(F(t),F[t];\Phi)/\mathcal{M} \;\cong\; \begin{pmatrix} F(t) & 0 \\ Ry/I & 0 \end{pmatrix},$$

for some left R-ideal $I \subseteq Ry$. Notice $te_{1,1}$ gives rise to an endomorphism of

$$\begin{pmatrix} F(t) & 0 \\ Ry/I & 0 \end{pmatrix}$$

by left multiplication. Moreover, this endomorphism is not algebraic over F, as t is not algebraic over F. It follows that $\mathcal{A}(F(t),F;\Phi)$ does not have the endomorphism property. In particular, $\mathcal{A}(F(t),F;\Phi)$ does not satisfy the Nullstellensatz. We now prove a proposition to show that $\mathcal{A}(F(t),F;\Phi)$ is Jacobson.

Proposition 3.4 *$\mathcal{A}(F(t),F;\Phi)$ is Jacobson.*

Proof. Let I be a nonzero prime ideal of $\mathcal{A}(F(t),F;\Phi)$. Since $\mathcal{A}(F(t),F;\Phi)$ is prime, $\overline{e_{1,1}}I\overline{e_{1,1}}$ is a nonzero ideal of $\overline{e_{1,1}}\mathcal{A}(F(t),F;\Phi)\overline{e_{1,1}} \cong F(t)$. Hence

$$\overline{e_{1,1}} \in \overline{e_{1,1}}I\overline{e_{1,1}}.$$

It follows that

$$Ry\overline{e_{2,1}} = (Ry\overline{e_{2,1}})\overline{e_{1,1}} \in I,$$

$$R\overline{e_{1,2}} = \overline{e_{1,1}}(R\overline{e_{1,2}}) \in I,$$

and

$$RyR\overline{e_{2,2}} = (Ry\overline{e_{2,1}})\overline{e_{1,1}}(R\overline{e_{1,2}}) \in I.$$

Let

$$(p(x)\overline{e_{2,2}}) \;=\; \overline{e_{2,2}}I\overline{e_{2,2}} \cap F[x]\overline{e_{2,2}}.$$

We have

$$\mathcal{A}(F(t), F; \Phi)/I \cong \begin{pmatrix} 0 & 0 \\ 0 & F[x]/(p(x)) \end{pmatrix} \cong F[x]/(p(x)).$$

Since $F[x]$ is Jacobson, we conclude that

$$\mathcal{A}(F(t), F; \Phi)/I$$

has zero Jacobson radical for every nonzero prime ideal I. Since $\mathcal{A}(F(t), F; \Phi)$ is a primitive ring, we conclude that it is Jacobson. ∎

Using different methods, Irving [6] was able to construct for any $d \geq 1$ an example of an affine F-algebra A of GK dimension $d + 2$ with

$$\mathrm{End}_A(M) \cong F(t_1, \ldots t_d)$$

for some simple module M.

Corollary 9.1.8 of McConnell and Robson [9] shows that when F is uncountable, any countably generated F-algebra satisfies the Nullstellensatz. Also, a primitive ring of GK dimension one is PI and hence satisfies the Nullstellensatz by Theorem 13.10.3 of McConnell and Robson [9]. We ask the following question.

Question 3.3 *Suppose R is an affine primitive Jacobson F-algebra of GK dimension less than three. Then does R necessarily satisfy the Nullstellensatz over F?*

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