MATH 245: Assignment 2, Due Sept. 21, 2018 (to be submitted via crowdmark)

(1) Let F be the field with p elements, p a prime. Recall that an $n \times n$ matrix A in $M_n(F)$ is invertible if and only if its columns $v_1, \ldots, v_n$ form a basis for F^n. Use this to show that the number of $n \times n$ invertible matrices in $M_n(F)$ is the same as the number of ordered bases for F^n, which is the same as the number of sequences $v_1, \ldots, v_n$ such that $v_1, \ldots, v_i$ is linearly independent for $i = 1, \ldots, n$. Now let's count! Show that we have $p^n - 1$ choices for the first vector v_1. In general, show that if you have picked $v_1, \ldots, v_i$ so that they are linearly independent then there are p^i elements in their span and the only constraint on picking v_{i+1} is that it cannot be in the span of $v_1, \ldots, v_i$. How many choices do you have for v_{i+1}. Put it together to get a formula for the number of invertible matrices in $M_n(F)$.

(2) Let $X_1, \ldots, X_{n^2}$ be a basis for $n \times n$ matrices over a field F. Suppose that Y is an $n \times n$ matrix with entries in F such that $\mathrm{Trace}(YX_i) = 0$ for $i = 1, \ldots, n^2$. Show that Y is the zero matrix.

(3) Let A be an $n \times n$ integer matrix and suppose that $A^s = I$ for some $s \geq 1$. Show that the trace of A is in the set

$$\{-n, -n+1, \ldots, 0, 1, \ldots, n\}.$$

(4) Suppose that $S \subseteq M_n(\mathbb{Z})$ is a set of matrices with the following properties: (1) S contains a basis for $M_n(\mathbb{Q})$; (2) every element A in S satisfies $A^s = I$ for some $s \geq 1$, depending upon A; (3) if $A, B \in S$ then $A \cdot B \in S$. Show that S is finite and in fact has size at most $(2n+1)^{n^2}$.

(5) Let p be prime. Show that if $0 \leq i, j < p$ and $0 \leq a < b$ are integers then the binomial coefficients satisfy

$$\binom{pb+j}{pa+i} \equiv \binom{b}{a} \cdot \binom{j}{i} \pmod{p},$$

where we take $\binom{i}{j}$ to be zero if $i < j$ and we take $\binom{0}{0} = 1$, which aligns with the usual convention that an empty product is 1. Use this to show by induction that $\binom{p^d}{i} \equiv 0 \ (\mathrm{mod} p)$ for $i = 1, 2, \ldots, p^d - 1$.

(6) A matrix N is called *nilpotent* if $N^d = 0$ for some $d \geq 1$. Show that if F is a field of characteristic p, with p a prime, then if N is an $n \times n$ nilpotent matrix in $M_n(F)$ then $(I+N)^{p^d} = I$ whenever $p^d \geq n$.

(7) Let H_n denote the $n \times n$ matrix whose (i,j)-entry is $1/(i+j-1)$. Show that if $w = [w_0, w_1, \ldots, w_{n-1}]^T, v = [v_0, v_1, \ldots, v_{n-1}]^T \in \mathbb{R}^n = \mathbb{R}^{n \times 1}$ then $w^T H_n v$ is equal to

$$\int_0^1 p(t)q(t)\, dt$$

where $p(t) = v_0 + v_1 t + \cdots + v_{n-1}t^{n-1}$ and $q(t) = w_0 + w_1 t + \cdots + w_{n-1}t^{n-1}$.

(8) Show that $v^T H_n v$ is strictly positive whenever v is a nonzero vector in $\mathbb{R}^n$ and that H_n is an invertible matrix.

(9) Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear operator. Suppose that $V = W_1 \oplus \cdots \oplus W_d$ where each W_i is T-invariant. Show that there is a basis B for V such that we obtain a block-diagonal matrix

$$[T]_B = \begin{pmatrix} A_1 & 0 & 0 & \ldots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{d-1} & 0 \\ 0 & 0 & \cdots & 0 & A_d \end{pmatrix},$$

where A_i is the matrix of the restriction operator $T|_{W_i} : W_i \to W_i$ with respect to some basis for W_i. Show also that the minimal polynomial of T is the least common multiple of the minimal polynomials of the matrices $A_1, A_2, \ldots, A_d$ and that the characteristic polynomial of T is the product of the characteristic polynomials of the A_i.

(10) Let T be a linear operator on a finite-dimensional vector space V. We saw in class that for a vector v, the T-annihilating polynomial, $p_v(T)$ always divides the minimal polynomial. Show that the minimal polynomial is in fact the least common multiple of the polynomials $p_v(T)$ as v ranges over all vectors in V.