MATH 245: Assignment 3, Due Sept. 28, 2018 (to be submitted via crowdmark)

(1) Let V denote the $\mathbb{R}$-vector space consisting of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Let $\lambda_1, \ldots, \lambda_s$ be distinct real numbers. Show that $e^{\lambda_1 x}, \ldots, e^{\lambda_s x}$ are linearly independent elements of V. (Hint: Argue via contradiction: assume that a dependence exists and take one with the fewest number of nonzero coefficients. Now differentiate your equation with respect to x and see if you can get a smaller dependence!)

(2) Let V be as in the preceding question. Elements $f(x)$ and $g(x)$ of V are called *algebraically dependent* over $\mathbb{R}$, if there is a nonzero polynomial $P(u,v) \in \mathbb{R}[u,v]$ such that $P(f(x), g(x))$ is identically zero. If $f(x)$ and $g(x)$ are not algebraically dependent over $\mathbb{R}$, we say they are algebraically independent over $\mathbb{R}$. Show that e^{2x} and e^{3x} are algebraically dependent over $\mathbb{R}$ but e^x and $e^{\sqrt{2}x}$ are algebraically independent over $\mathbb{R}$.

(3) We say that a subset I of $M_n(F)$ is an ideal if $0 \in I$, if $A, B \in I \implies A + B \in I$ and if $A \in I$ and $C \in M_n(F)$ then $CA \in I$ and $AC \in I$. Compare this with our definition of an ideal of $F[x]$. Notice we do the condition with left and right multiplication because matrix multiplication is not commutative. Show that the only ideals in $M_n(F)$ are $\{0\}$ and all of $M_n(F)$. (Hint: suppose that I is an ideal that contains a nonzero matrix A. Consider matrices of the form $E_{s,i}AE_{j,t}$. What do you get?)

(4) Let V be a vector space. We let $\mathrm{End}(V)$ denote the set of linear transformations $T : V \to V$. We say that a subset I of $\mathrm{End}(V)$ is an ideal if the zero transformation is in I and if $T_1, T_2 \in I$ then $T_1 + T_2 \in I$ and if $T \in I$ and $S \in \mathrm{End}(V)$, then $T \circ S$ and $S \circ T$ are both in I. Show that if V is finite-dimensional then the only ideals of $\mathrm{End}(V)$ are $\{0\}$ and $\mathrm{End}(V)$ but show that if V is infinite-dimensional that $I := \{T : V \to V : \mathrm{im}(T) \text{ is finite} - \text{dimensional}\}$ is a proper nonzero ideal of $\mathrm{End}(V)$, where $\mathrm{im}(T) = T(V)$ is the image of T.

(5) Let V be an F-vector space and let W be a subspace of V. Recall that V/W is the collection of equivalence classes $[v] = \{v' : v \sim v'\}$ where $\sim$ is our equivalence relation and $v, v' \in V$ are equivalent, that is, $v \sim v'$, if $v - v' \in W$. Show that V/W is an F-vector space with addition defined by $[v_1] + [v_2] = [v_1 + v_2]$ and scalar multiplication given by $c \cdot [v] = [cv]$. (Hint: when you deal with equivalence classes, you have to show things are well defined first).

(6) Show that if $T : V \to V$ is a linear transformation and W is a T-invariant transformation then we have a linear transformation $\bar{T} : V/W \to V/W$ defined by $\bar{T}([v]) := [Tv]$. (Hint: again, you have to show everything is well defined!)

(7) Show that if V is a vector space and W is a subspace of V then we have a canonical surjective linear transformation $\pi : V \to V/W$ defined by $\pi(v) = [v]$ and that the kernel of π is equal to W.

(8) Let V be a finite-dimensional vector space and let W be a subspace of V. Show that the dimension of the space V/W is equal to $\dim(V) - \dim(W)$.

(9) Let $T : V \to V$ be a linear transformation and let W denote the set of vectors that are annihilated by some power of T. Then W is T-invariant and if we let $\bar{T}$ denote the linear transformation from V/W to itself defined in question 6, then show that $\bar{T} : V/W \to V/W$ is an isomorphism.

(10) Let A, B, C be 2×2 matrices with entries in a field F. Show that $(AB - BA)^2C - C(AB - BA)^2 = 0$. (Hint: you can do this two ways. One is computationally intensive and involves working with 12 unknowns and calculating, and you will be mad at me when you are done. The other involves the Cayley-Hamilton theorem and you'll be happy with me when you are done. Show that the Cayley-Hamilton theorem for 2×2 matrices T says that

$$T^2 - \mathrm{Tr}(T)T + \det(T)\mathrm{id} = 0.$$

So what if you take $T = AB - BA$?)