MATH 245: ASSIGNMENT 1, DUE SEPT. 14, 2018 (TO BE SUBMITTED VIA CROWDMARK)

(1) (Useful basic fact that everyone should know) Let $E_{i,j}$ denote the matrix whose (i,j)-entry is 1 and all other entries are 0. Show that $M_n(F)$ is an F-vector space with basis $\{E_{i,j} \colon 1 \le i,j \le n\}$ and $E_{i,j}E_{s,t} = \delta_{j,s}E_{i,t}$ where $\delta_{j,s} = 1$ if $j = s$ and is zero otherwise. Use this to show that if a matrix commutes with every matrix, then in particular it must commute with every $E_{i,j}$, and then show it must be a scalar multiple of the identity.

(2) Let A be the companion matrix of the polynomial $x^2 - x - 1$. Guess and prove a formula for A^n in terms of well known mathematical sequences.

(3) Let V be a finite-dimensional F-vector space and let T be a linear operator on V. We call the *centralizer* of T, which we denote $C(T)$, the set of linear operators $U : V \to V$ such that T and U commute (i.e., $T \circ U = U \circ T$). Show that the centralizer is closed under addition, scalar multiplication (i.e., if $\lambda \in F$, and $U \in C(T)$, then $\lambda U \in C(T)$) and under composition.

(4) Suppose that every linear operator $U : V \to V$ is in $C(T)$ for some linear operator $T : V \to V$. Show that there is some $c \in F$ such that $T(v) = cv$ for every $v \in V$ and that the converse holds.

(5) Let T be a linear operator on a vector space V. Show that every polynomial in T, $c_0 + c_1T + \cdots + c_dT^d$, is in $C(T)$.

(6) Let T be a linear operator on a finite-dimensional space V. Recall that a vector v is a cyclic vector (if it exists) for a linear operator T if $v, Tv, T^2v, \ldots, T^{n-1}v$ is a basis for V, where n is the dimension of V. Prove that T has a cyclic vector if and only if every linear operator U on V that commutes with T is a polynomial in T.

(7) (Very Useful Fact for Linear Algebra) Let F be a field and let $x_1, \ldots, x_n$ be variables. We let $F[x_1, \ldots, x_n]$ denote the set of finite F-linear combinations of monomials $x_1^{i_1} \cdots x_n^{i_n}$ with multiplication given by $cx_1^{i_1} \cdots x_n^{i_n} \cdot c'x_1^{j_1} \cdots x_n^{j_n} = cc'x_1^{i_1+j_1} \cdots x_n^{i_n+j_n}$, and then extending by linearity. Notice that if

$$P(x_1, \ldots, x_n) = \sum c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

is a polynomial (so $c_{i_1,\ldots,i_n} = 0$ for all but finitely many $(i_1, \ldots, i_n)$) and $c_1, \ldots, c_n \in F$ then we can evaluatie P at $(c_1, \ldots, c_n)$ via the rule

$$P(c_1, \ldots, c_n) = \sum c_{i_1,\ldots,i_n} c_1^{i_1} \cdots c_n^{i_n} \in F$$

Show that if F is an infinite field and $p(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ is a nonzero polynomial then there exist $c_1, \ldots, c_n \in F$ such that $P(c_1, \ldots, c_n) \ne 0$. (Hint: try induction!)

(8) Let F be a field and let K be a field that contains F. Suppose that $A_1, \ldots, A_d \in M_n(F)$ have the property that $c_1A_1 + \cdots + c_dA_d$ is invertible for some $c_1, \ldots, c_d \in K$. Show that if F is infinite then there exist $b_1, \ldots, b_d \in F$ such that $b_1A_1 + \cdots + b_dA_d$ is invertible.

(9) Prove that the above does not necessarily hold if F is a finite field.

(10) Prove that if F is an infinite field then every matrix in $M_n(F)$ can be written as a sum of two invertible matrices. (Hint: Try to see if you can use the Very Useful Fact.)

(11) Let F be the field with 5 elements. How many invertible 2×2 matrices in $M_2(F)$ are there? How many invertible 3×3 matrices in $M_3(F)$ are there?