Hilbert basis theorem: the argument

Let's give the argument in the Hilbert basis theorem in steps we can understand. Let R be a noetherian ring. Recall that given $p(x) \in R[x]$, we let $\deg(p(x))$ denote its degree and we let $\text{in}(p(x))$ denote the coefficient of the highest power of x that occurs in $p(x)$ with nonzero coefficient. So $\text{in}(p(x)) \in R$.

KEY STEPS IN THE ARGUMENT

(1) We start with a nonzero ideal I in $R[x]$. Our goal is to show that I is finitely generated. Our first step is that we take f_1 in $I \setminus 0$ of smallest degree. We let $I_1 = f_1 R[x]$ and we let $a_1 = \text{in}(f_1)$ and $J_1 = a_1 R$.

(2) At step $n+1$, having found $f_1, \ldots, f_n$ in I and $a_1, \ldots, a_n \in R$, and ideals $I_n = f_1 R[x] + \cdots + f_n R[x]$ and $J_n = a_1 R + \cdots + a_n R$, we pick $f_{n+1} \in I \setminus I_n$ of minimal degree (if we cannot do this, we stop and we have $I = I_n$ and so I is f.g.). We let $a_{n+1} = \text{in}(f_{n+1})$ and we let $I_{n+1} = I_n + f_{n+1} R[x]$ and $J_{n+1} = J_n + a_n R$.

(3) Since R is noetherian, there is some n such that

$$J_n = J_{n+1} = \cdots .$$

We claim that $I = I_n$.

(4) To see this, if it is not the case then the algorithm does not terminate at the $n+1$-st step and so $f_{n+1} \in I \setminus I_n$.

(5) By minimality, we have that the degree of f_{n+1} is at least as big as the degrees of $f_1, \ldots, f_n$. Also, since $a_{n+1} = \text{in}(f_{n+1}) \in J_{n+1} = J_n$, we have $a_{n+1} = r_1 a_1 + \cdots + r_n a_n$ for some $r_1, \ldots, r_n \in R$.

(6) Let $d_i = \deg(f_i)$ for $i = 1, \ldots, n+1$. Since $\deg(f_{n+1}) \geq \deg(f_i)$ for $i \leq n$, we have

$$h := f_{n+1} - r_1 x^{d_{n+1}-d_1} f_1 - \cdots - r_n x^{d_{n+1}-d_n} f_n$$

is in I and has degree strictly less than d_{n+1}. So by minimality of $\deg(f_{n+1})$ we must have that $h \in I_n$.

(7) Now what? if $h \in I_n$ then so is f_{n+1} since

$$f_{n+1} = h + r_1 x^{d_{n+1}-d_1} f_1 + \cdots + r_n x^{d_{n+1}-d_n} f_n$$

and every term in the RHS is in I_n. This is a contradiction!

(8) Conclusion $I = I_n$ as claimed and so I is generated by $f_1, \ldots, f_n$. Since every ideal is f.g., we see that $R[x]$ is noetherian.