Assignment 4. Due November 24 in class. Feel free to assume the results from any earlier questions in doing the later questions. Throughout all of these exercises all varieties are over an algebraically closed field. (Warning: this assignment is hard.)

(1) Show that

$$\Delta^j(f(z)) = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} f(z+i).$$

(2) Prove the following (not easy) folklore result (apparently not written down in the literature). Let X be an irreducible quasi-projective variety over an uncountable algebraically closed field k. Show that X cannot be written as a countable union of proper closed subsets. (Hint: This is a bit like the Baire category theorem. First write X as an open subset of a projective subvariety Y. Show that Y is irreducible and that it is sufficient to show that Y cannot be written as a union of proper closed subsets. Next use the fact that there is some n such that Y is closed in $\mathbb{P}^n$. Use the affine cover of $\mathbb{P}^n$ to write Y as a union of affine open subsets. Show that if Y can be written as a countable union then so can these affine subsets and show that one can reduce to the case where X is a closed irreducible subset of some $\mathbb{A}^n$. Now let $I = I(X)$ be the ideal of X and let $R = k[x_1, \ldots, x_n]/I$. Then I is a prime ideal since X is irreducible and so R is an integral domain. Then R is the affine coordinate ring of A. Now show that if $Y_1, Y_2, \ldots$ are proper closed subsets of X then there are polynomials $f_1, f_2, \ldots$ in $k[x_1, \ldots, x_n]$ such that $Y_i \subseteq V(f_i = 0) \cap X$ and such that $f_i \notin I$. Now let S denote the multiplicative set generated by $f_1, f_2, \ldots$; that is, S is all monomials in $f_1, f_2, \ldots$. Then let $A = S^{-1}R$. Show that A is countably infinite dimensional as a k-vector space. Let P be a maximal ideal of A. Show that $A/P = k$ (under the natural identification of k with the image of k under the composition $k \to A \to A/P$). (Why? A/P is countably infinite dimensional and it is a field extension of k—since k is uncountable and algebraically closed this forces A/P to be k.) So what does this mean? This means that if Q is the kernel of the composition $R \to A \to A/P$ then $R/Q = k$ and so Q corresponds to an ideal $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ in $k[x_1, \ldots, x_n]$ that contains I. That means that $(\alpha_1, \ldots, \alpha_n)$ is a point on X that is not in any of the Y_i. Think about all of this!)

(3) Show that the previous result need not hold if k is countable and algebraically closed.

(4) A map $\sigma : X \to X$ is said to preserve a non-constant fibration if there is a non-constant rational function $h \in k(X) = \mathcal{O}_{X,X}$ such that $\sigma^*(h) := h \circ \sigma = h$. Show that σ^n preserves a non-constant fibration if and only if σ does. (Hint: one direction is trivial. For the other direction, suppose that $(\sigma^n)^*(h) = h$ for some non-constant rational function h. Let $h_i = (\sigma^i)^*(h)$ for $i = 0, \ldots, n-1$. Let θ_i denote the coefficient of x^i in $(x - h_0)(x - h_1) \cdots (x - h_n)$. Show that each θ_i is a rational function on X such that $\sigma^*(\theta_i) = \theta_i$. Show that if they are all constant then the h_i must be constant and so h must be too, contradiction.)

(5) Let $\sigma : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ be an automorphism of $\mathbb{A}^2_{\mathbb{C}}$. (That is, $\sigma(x,y) = (P(x,y), Q(x,y))$ for some polynomials P, Q and we have a polynomial inverse.) Show that if there does not exist a rational function $f(x,y)$ such that $\sigma^*(f) := f \circ \sigma = f$ then there are at most two irreducible curves $C \subseteq \mathbb{A}^2$ such that $\sigma^n(C) = C$ for some $n > 0$. (In geometric language, people say that if σ does not preserve a non-constant fibration then there are only finitely many σ-periodic curves.) (Hint: this is fun. An irreducible curve in $\mathbb{A}^2$ is the zero set of an irreducible polynomial $f(x,y) \in \mathbb{C}[x,y]$. Suppose that $C = V(f(x,y) = 0)$ and $\sigma^n(C) = C$. This means that $f(a,b) = 0$ if and only if $(\sigma^n)^*(f)(a,b) = f(\sigma^n(a,b)) = 0$. Show that this occurs if and only if the polynomial $f \circ \sigma^n = \lambda f$ for some nonzero scalar λ—you probably need to use the fact that σ is an automorphism here. Now what? Suppose we have 3 distinct irreducible curves $V(f_1 = 0)$, $V(f_2 = 0)$, $V(f_3 = 0)$ that are periodic under σ. Show that there is some common n and nonzero scalars $\lambda_1, \lambda_2, \lambda_3$ such that $f_i \circ \sigma^n = \lambda_i f_i$. Now the field of rational functions $\mathbb{C}(x,y)$ has transcendence degree 2 and so f_1, f_2, f_3 are algebraically dependent. Let

$$\sum_{i,j,k} c_{i,j,k} f_1^i f_2^j f_3^k = 0$$

be a non-trivial dependence that is minimal with respect to having the smallest number of monomials appearing with a nonzero coefficient. Show that we must have at least two nonzero coefficients. Apply $(\sigma^n)^*$ to both sides to get

$$\sum_{i,j,k} c_{i,j,k}\lambda_1^i\lambda_2^j\lambda_3^k f_1^i f_2^j f_3^k = 0.$$

Use minimality to show that $\lambda_1^i\lambda_2^j\lambda_3^k$ must be constant as (i,j,k) runs over all 3-tuples with $c_{i,j,k}$ nonzero. Now show that if (i,j,k) and (i',j',k') are two such triples then $f_1^{i-i'}f_2^{j-j'}f_3^{k-k'}$ is fixed by $(\sigma^n)^*$. Why is it non-constant? Now you're almost done.)

(6) Let $\sigma : X \to X$ be an endomorphism of an irreducible variety X. A point $x \in X$ is said to be σ-periodic if there is some $n \geq 1$ such that $\sigma^n(x) = x$. It is σ-preperiodic if there exist $m < n$ such that $\sigma^m(x) = \sigma^n(x)$. Give an example of an endomorphism σ of some variety and a point that is σ-preperiodic but not σ-periodic.

(7) Let $\sigma : X \to X$ be an endomorphism of a quasi-projective irreducible variety X. Show that the collection of σ-perperiodic points is a countable union of proper closed subvarieties unless there exist $m < n$ such that $\sigma^m = \sigma^n$. (Hint: you may assume the well-known fact that the diagonal of $X \times X$ is closed.)

(8) Let $\sigma : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ be an automorphism of $\mathbb{A}^2_{\mathbb{C}}$. (That is, $\sigma(x,y) = (P(x,y),Q(x,y))$ for some polynomials P,Q and we have a polynomial inverse.) Show that if there does not exist a rational function $f(x,y)$ such that $f \circ \sigma = f$ then there is a point $(\alpha,\beta) \in \mathbb{A}^2$ that has a Zariski dense orbit under σ. (The orbit of a point a under σ is just the set $\{a, \sigma(a), \sigma^2(a), \ldots\}$ (Hint: show that if a point does not have a dense orbit, then the closure is the union of a (possibly empty) finite set of periodic points and a finite union of σ-periodic curves. Next use earlier examples to show that the union of all periodic points and all periodic curves is a countable union of proper closed sets. Use Q2 to show that there is some point not in this countable union and show that such a point must have a dense orbit.)

(9) Let $\sigma : \mathbb{A}^2 \to \mathbb{A}^2$ be an automorphism (over $\mathbb{C}$) and suppose that $\sigma^n(\alpha,\beta) \in \mathbb{A}^2_{\mathbb{R}}$ for infinitely many n. Show that it holds on an arithmetic progression.