Assignment 3. Due November 3 in class. Feel free to assume the results from any earlier questions in doing the later questions.

(1) A topological space is said to be *Noetherian* if every descending chain of closed subsets terminates. Show that every subset of a Noetherian space is compact (when given the subspace topology).

(2) Show that if X is a topological space with every subset compact (with subspace topology) then X is noetherian.

(3) Given an associative (but not necessarily commutative) ring R with 1 we let $\mathrm{Spec}(R)$ denote the set of prime ideals of R. (An ideal P of R is called prime if whenever $a, b \in R$ have the property that $arb \in P$ for all $r \in R$ we must have $a \in P$ or $b \in P$. We put a topology on $\mathrm{Spec}(R)$ by declaring the closed sets to be precisely the sets of the form $C_I = \{P \in \mathrm{Spec}(R) \colon P \supseteq I\}$, where I is a two-sided ideal of R. Show that this does indeed give a topology on R, which we call the *Zariski topology.* (Hint: if you understand the commutative case, you probably understand this case too.)

(4) Show that if R is Noetherian as a ring then $\mathrm{Spec}(R)$ is Noetherian as a topological space.

(5) Give an example of a non-Noetherian ring R with $\mathrm{Spec}(R)$ Noetherian. (Hint: you can do it so that $\mathrm{Spec}(R)$ is a point.)

(6) (Optional) A k-algebra R is said to be $\mathbb{N}$-*graded* if R has a decomposition $R = \bigoplus_{n=0}^{\infty} R_n$ where each R_i is a k-vector subspace of R and $R_i R_j \subseteq R_{i+j}$ for $i, j \geq 0$. R is called connected graded if $R_0 = k$. An element $r \in R$ is called homogeneous if there is some n such that $r \in R_n$ and an ideal I of R is called homogeneous if I is generated by homogeneous elements. Given a connected graded ring R, we let $\mathrm{Proj}(R)$ denote the set of homogeneous prime ideals that do not contain the positive part with the subspace topology (regarding Proj as a subset of Spec). Is there an example of a connected graded commutative domain R where $\mathrm{Proj}(R)$ is Noetherian but $\mathrm{Spec}(R)$ is not?

(7) We'll say that a topological space is Artinian if every ascending chain of closed subsets terminates. Show that if X is an Artinian space then every $x \in X$ has a unique smallest open neighbourhood U_x of x.

(8) Show that a Hausdorff Artinian space is discrete and finite.

(9) Let X be a Noetherian topological space and let $\phi : X \to X$ be a continuous map and let Y be a closed subset and let $x \in X$. Show that if $\phi^n(x) \in Y$ for all composite numbers n then $\phi^n(x) \in Y$ for all sufficiently large natural numbers n.

(10) Give an example that shows the conclusion to question 9 need not hold if X is not noetherian.

(11) A closed subset Y of X is irreducible if Y cannot be written as a union of two proper closed subsets. We define the dimension of a Noetherian space X to be the supremum of the lengths of descending chains of irreducible closed subsets of X. Show that $X = [0, 1]$ with closed subsets being the empty set, X, and sets of the form $[1/n, 1]$ with $n \geq 1$ gives an infinite dimensional noetherian topology on X.

(12) Let $F(x) \in \mathbb{C}[[x]]$ be an algebraic power series of degree 2; that is we have

$$\sum_{i=0}^{2} p_i(x) F(x)^i = 0$$

for some complex polynomials $p_i(x)$, with $p_2 p_0$ nonzero. Show that $F(x)$ satisfies a differential equation of the form

$$\sum_{i=0}^{m} q_j(x) F^{(j)}(x)$$

where the $q_j(x)$ are polynomials, not all zero. Hint: consider the equation $F^2 + bF + c = 0$ with b, c rational functions in x. Differentiate to get

$$2FF' + b'F + bF' + c' = 0.$$

Now try multiplying this by $(F + b)$ and see where you can go from there. (In fact, this fact doesn't need degree 2—it is true for any algebraic power series, but the proof is a bit more involved.)