Assignment 2. Due October 15 in class. Feel free to assume the results from any questions in doing the later questions.

(1) Show that $|\mathrm{Re}((1+\sqrt{-2})^n)| \to \infty$ as $n \to \infty$.

(2) Let Δ be a finite set and let $p \geq 2$ be a natural number. Show that a map $f : \mathbb{N}_0 \to \Delta$ is p-automatic if and only if the p-kernel of f is finite and show that if $g(n)$ is in the p-kernel of f then the p-kernel of $g(n)$ is contained in the p-kernel of $f(n)$.

(3) Prove the pumping lemma: If $\mathcal{L} \subseteq \Sigma^*$ is an infinite regular language then there exist words $a, b, c \in \Sigma^*$ with b having length at least one such that $ab^nc \in \mathcal{L}$ for all $n \geq 0$.

(4) Let Δ be a finite subset of a ring R and suppose that $f, g : \mathbb{N}_0 \to \Delta$ are p-automatic. Show that $f+g$ and fg are both p-automatic (now the ranges are respectively $\Delta + \Delta$ and Δ^2).

(5) Let $p \geq 2$ and let S and T be p-automatic sets and let a and b be nonnegative integers with $a \geq 1$. Show that $aS + b = \{an + b \colon n \in S\}$ is p-automatic and that $S \cup T$ and $S \cap T$ are p-automatic.

(6) Let K be a finitely generated field extension of $\mathbb{F}_p$ and let $K^{\langle p \rangle} = \{x^p \colon x \in K\}$. Show that $K^{\langle p \rangle}$ is a field and that K is finite-dimensional as a $K^{\langle p \rangle}$-vector space. Show that K can be finite-dimensional or infinite-dimensional if K is not finitely generated.

(7) Let $\Sigma = \{0, 1, \ldots, p-1\}$ and let $w_0, w_1, \ldots, w_m, t_1, \ldots, t_m \in \Sigma^*$. Show that there exist rational numbers $c_0, c_1, \ldots, c_m$ and nonnegative integers $s_1, \ldots, s_m$ such that

$$\{[w_0 t_1^{i_1} w_1 t_2^{i_2} \cdots w_{m-1} t_m^{i_m} w_m]_p \colon i_1, i_2, \ldots, i_m \geq 0\}$$

is exactly the set of natural numbers from

$$\{c_0 + c_1 p^{s_1 j_1} + c_2 p^{s_1 j_1 + s_2 j_2} + \cdots + c_m p^{s_1 j_1 + \cdots + s_m j_m} \colon j_1, \ldots, j_m \geq 0\}.$$

(8) Show directly that the set $S = \{1, 3, 3^2, 3^3, \ldots\}$ is p-automatic if and only if p is a power of 3.

(9) Show that if $\mathcal{L}$ is a regular language then if $f(n)$ is the number of words in $\mathcal{L}$ of length n, then $f(n)$ satisfies a linear recurrence over $\mathbb{Q}$.

(10) Let K be a field. We say that a power series $F(x) \in K[[x]]$ is algebraic (over $K(x)$) if there exist $m \geq 1$ and polynomials $P_0(x), \ldots, P_m(x) \in K[x]$, not all zero, such that

$$\sum_{i=0}^{m} P_i(x) F(x)^i = 0.$$

Notice that power series expansions of rational functions (consider just ones that do not have a pole at $x = 0$) are all algebraic where we can take $m = 1$. Show that if q is a power of a prime p and $f : \mathbb{N}_0 \to \mathbb{F}_q$ is p-automatic then

$$F(x) := \sum_{n \geq 0} f(n) x^n \in \mathbb{F}_q[[x]]$$

is algebraic. (Hint: this isn't so easy. Using exercise 2, we know that the p-kernel of $f(n)$ is finite. Let $f(n) = f_1(n), \ldots, f_r(n)$ denote the distinct sequences in the p-kernel and let $F_i(x) = \sum_{n \geq 0} f_i(n) x^n$. Then $F(x) = F_1(x)$. Now for each i show that

$$F_i(x) = \sum_{j=0}^{q-1} \sum_{n=0}^{\infty} f_i(qn + j) x^{qn+j}.$$

By the second part of Q2, we have that $f_i(pn + j)$ is in the p-kernel of f and hence is equal to some $f_{m_{i,j}}(n)$. From this show that

$$F_i(x) = \sum_{j=0}^{q-1} x^j F_{m_{i,j}}(x)^q$$

(Here you should use the fact that $a^q = a$ for $a \in \mathbb{F}_q$.) Conclude that $F_i(x)$ is in the $\mathbb{F}_q(x)$-vector subspace of $\mathbb{F}_q((x))$ (the field of fractions of $\mathbb{F}_q[[x]]$) spanned by $F_1(x)^q, \ldots, F_r(x)^q$. Show by induction that for each m and each $j \leq m$ we have that $F_i(x)^{q^j}$ is in the $\mathbb{F}_q(x)$-vector subspace of $\mathbb{F}_q((x))$ spanned by $F_1(x)^{q^m}, \ldots, F_r(x)^{q^m}$. Now argue by dimensions that if we pick $m > r$ then $F_1(x), F_1(x)^q, \ldots, F_1(x)^{q^{m-1}}$ must have a non-trivial linear dependence over $\mathbb{F}_q(x)$.)

(11) In fact the converse holds—this is the well-known Christol's theorem. It says that $F(x) \in \mathbb{F}_q[[x]]$, q a power of p, is algebraic if and only if its sequence of coefficients is p-automatic. Use Christol's theorem to show that $\sum_{j=0}^{\infty} x^{3^j} \in \mathbb{Q}[[x]]$ is not algebraic over $\mathbb{Q}(x)$.

(12) Use the pumping lemma to prove the following theorem of Minsky and Pappert: the set of prime numbers is not a k-automatic set for any $k \geq 2$. (Hint: if there were some k for which it were, then we would have some input alphabet Σ and words $a, b, c \in \Sigma^*$ with b of length ≥ 1 such that $[ab^n c]_k$ is always a prime number. Now show that there is some prime q such that $[ab^i c]_k = q$ with $q > k^{i \cdot \text{length}(b)}$ and show that $[ab^{iq}c]_k \equiv [ab^i c]_k \equiv 0 \pmod{q}$ and conclude that $[ab^{iq}c]_k = q$. Why is this a problem?)