Assignment 1. Due September 29 in class. Feel free to assume the results from any earlier questions in doing the later questions.

(1) Show that the inverse limit of an inverse system of nonempty groups is nonempty and forms a group.
(2) Show that an infinite profinite group is uncountable.
(3) Recall that a topological space X is totally disconnected if the connected components of X are the one-element subsets of X. Show that an inverse limit of totally disconnected topological groups is totally disconnected. Conclude that $\mathbb{Z}_p$ is totally disconnected.
(4) (For the next few questions use the analogy with vector spaces having a basis and try to adapt the proofs here to this setting—it might be a worthwhile exercise to study the corresponding arguments for vector spaces and see if you can adapt them here.) Recall that if a field K is an extension of a field k then we say that a subset S is algebraically independent if whenever $m \geq 0$ and $s_1, \ldots, s_m$ are distinct elements of S then if $p(s_1, \ldots, s_m) = 0$ for some polynomial $p(x_1, \ldots, x_m) \in k[x_1, \ldots, x_m]$ then $p(x_1, \ldots, x_m) = 0$. A subset is called a transcendence base if it is a maximally algebraically independent subset. Let K be a field extension of the field k. Show that K has a transcendence base over k.
(5) Show that any two transcendence bases for K/k have the same cardinality. This cardinality is called the *transcendence degree*. (Hint: look at the proof that bases for a vector space have the same cardinality and add a few new ideas.)
(6) Let K be an extension field of the field k and suppose that $\dim_k K$ is uncountable and k is countable. Show that K has uncountable transcendence degree over k.
(7) Show that $\dim_{\mathbb{Q}} \mathbb{Q}_p$ is uncountable and conclude that $\mathbb{Q}_p$ has uncountable transcendence degree over $\mathbb{Q}$.
(8) Urysohn's metrization theorem gives that a second-countable compact Hausdorff space is metrizable. (Recall that a topological space is second-countable if there exists a countable collection of open subsets such that every open set can be written as a union of elements from this countable collection.) Suppose that $(S, \leq)$ is a countable directed poset and $\widehat{G}$ is the inverse limit of an inverse system $\{G_s\}_{s \in S}$ of finite groups (endowed with the discrete topology). Show that $\widehat{G}$ (the inverse limit) is metrizable. (See more on this in questions 11 and 12.)
(9) Show that $\mathbb{Z}_p$ has a unique maximal ideal (recall that a ring having this property is called a local ring) given by $p\mathbb{Z}_p$. (Remember the old trick (well, not much of a trick) to show a maximal ideal is unique: show everything outside the set is a unit.)
(10) Show that every nonzero element of $\mathbb{Q}_p$ can be written uniquely as $p^m y$ where $m \in \mathbb{Z}$ and y is in $\mathbb{Z}_p^*$.
(11) Show that we can put a metric $|\cdot|_p$ on $\mathbb{Q}_p$ in which we define $|x|_p = p^{-m}$ if x is nonzero, where we write $x = p^m y$ with $m \in \mathbb{Z}$ and y a p-adic unit (that is a unit in $\mathbb{Z}_p$). (We take $|0|_p = 0$.)
(12) Show that the restriction of this metric to $\mathbb{Z}_p$ induces the same topology on $\mathbb{Z}_p$ as the profinite topology. (This relates to question 8—in general, if one has a countable inverse system of finite groups endowed with the discrete topology, one has that it is metrizable and one can define a distance function in a similar way (it is not unique, of course).