Assignment 2: Due Feb. 28, 2019

The purpose of the first few exercises on this assignment is to show that subgroups of free groups are free (you can do exercises 9 and 10 first if you prefer). For exercises 1–8, we let F be a free group on a set X and we let W be a subgroup of F. We let $\{W_i\colon i \in I\}$ be the set of right cosets of W, so they are indexed by a set I and we assume our index set contains an element 1 and that $W_1 = W$. We recall that a right transversal to a subgroup is just a collection of right coset representatives (i.e., an element from each coset). We choose a right transversal to W with the representative of the coset W_i denoted $\overline{W_i}$, so $\overline{Wa}$ is the coset representative for Wa. We assume that our transversal has the property $\overline{W} = \overline{W_1} = 1$. To each element $x \in X$ and $i \in I$ we associate a symbol $y_{i,x}$ and we let G denote the free group on the set $Y := \{y_{i,x}\colon i \in I, x \in X\}$.

(1) Show that the map $y_{i,x} \mapsto \overline{W_i}x\overline{(W_ix)}^{-1}$ induces a homomorphism $\tau : G \to W$.

(2) To each $u \in F$ and each $i \in I$ we construct an element u^{W_i} of G as follows. We may assume that u is a word in $X \cup X^{-1}$ with no occurrences of xx^{-1} or $x^{-1}x$ with $x \in X$. Define $1^{W_i} = 1$, $x^{W_i} = y_{i,x}$ for $x \in X$, and $(x^{-1})^{W_i} = (x^{W_ix^{-1}})^{-1}$ for $x \in X$. Then in general we define the map inductively by saying that if $u = vx$ with $y \in X \cup X$ and v is a reduced word shorter than u with the last letter of v not being y^{-1}, then we define $u^{W_i} := v^{W_i} \cdot y^{W_iv}$. Show, using induction on the length of v that under this assignment that for $u, v \in F$ we have $(uv)^{W_i} = u^{W_i}v^{W_iu}$ and $(u^{-1})^{W_i} = (u^{W_iu^{-1}})^{-1}$.

(3) Show by induction on the length of u that if $u \in F$ and $i \in I$ then $\tau(u^{W_i}) = \overline{W_i}u\overline{W_iu}^{-1}$

(4) Now we construct a map $\psi : W \to G$ by $u \mapsto u^W$. Show that ψ is a homomorphism and that $\tau(\psi(u)) = u$ for all $u \in W$ and conclude that ψ is one-to-one and τ is onto and $W \cong G/\ker(\tau)$.

(5) Let χ be the endomorphism of G given by $\chi = \psi \circ \chi : G \to G$. of G. Show that $\ker(\chi) = \ker(\tau)$ and so $W \cong G/\ker(\chi)$. Show that the kernel of χ is the normal closure, N, of the set $b_{i,x} := y_{i,x}^{-1}\chi(y_{i,x})$ with $i \in I$, $x \in X$. (Hint: show that each $b_{i,x}$ is in the kernel of χ and conclude that N is contained in $\ker(\chi)$. To do the other inclusion take an element g of G that is in the kernel of χ. Write g as a reduced word in the $y_{i,x}$. Use the fact to show that $\chi(y_{i,x}) \equiv y_{i,x} \pmod N$ and the fact that χ is a homomorphism to conclude that $\chi(g) \equiv g \pmod N$; that is $\chi(g)g^{-1} \in N$. But $\chi(g) = 1$ since g is in the kernel!

(6) Show in fact that the normal closure, K, of the elements u^W, with u a non-trivial element of our transversal, is equal to the kernel of τ.

(7) We're almost done! We say that a transversal Σ to W is a *Schreier transversal* if $v \in \Sigma$ whenever $vy \in \Sigma$ and $x \in X$ and vy is in reduced form (that is, the transversal is closed under taking prefixes of reduced words). Show that a Schreier transversal to W exists. (Hint: define the length of a coset Wa to be the minimum length of a word in Wa. So the only coset of length zero is $W = W_1$, which has the word empty-word 1. Suppose we have produced (reduced) coset representatives $\overline{W_i}$ for each coset W_i of length $< \ell$ in such a way that our set is closed under taking prefixes. Then if Wa is a coset of length ℓ and $u \in Wa$ has length ℓ, then we can write $u = vy$ with v of length $\ell - 1$ and $y \in X \cup X^{-1}$. Define $\overline{Wa} = \overline{Wv}y$ and show that this procedure inductively produces a Schreier transversal.)

(8) Now we'll prove that subgroups of free groups are free. Let Σ be a Schreier transversal to W in F and let $K = \ker(\tau) = \ker(\chi)$. Then from exercise 6, we have that K is the normal closure of u^W where u is a non-trivial word in the transversal. Show that if we have $u = vx^\epsilon$ with v of length $< u$ and $x \in X$ and $\epsilon \in \{\pm 1\}$ then $u^W = v^W y_{k,x}^\epsilon$ for some $k \in I$. Since v is in the transversal, conclude that K is the normal closure of these elements $y_{k,x}$ we obtain, which is a subset Y' of Y. Conclude that
$$W \cong G/K \cong \langle Y|Y'\rangle \cong F_{Y\setminus Y'}.$$

(9) Let F be a free group and let R be a subset of F. Show that the normal closure of R is the set of elements that can be expressed in the form $g_1r_1^{\epsilon_1}g_1^{-1}g_2r_2^{\epsilon_2}g_2^{-1}\cdots g_sr_s^{\epsilon_s}g_s^{-1}$ with $s \geq 1$, $\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}$, and $g_1, \ldots, g_s \in F$ and $r_1, \ldots, r_s \in R$.

(10) Let F be a free group on a set X and let $R = \{xyx^{-1}y^{-1}\colon x, y \in X\}$. Show that $\langle X|R\rangle \cong \mathbb{Z}^{\oplus X}$, the free abelian group on X. (Hint: let $G = \mathbb{Z}^{\oplus X}$ be the free abelian group with generators e_x, with $x \in X$. Show we have a homomorphism $\Phi : F \to G$ induced by the set map $x \mapsto e_x$ and that the kernel contains R and so the normal closure, N, of R is contained in the kernel of Φ. Hence Φ induces a surjective homomorphism $\Phi' : \langle X|R\rangle \to G$. Next show that $\langle X|R\rangle$ is abelian and generated by elements x and so we have a surjective homomorphism $\Psi : G \to \langle X|R\rangle$ that sends e_x to the coset xN. Show that $\Phi \circ \Psi : G \to G$ is an isomorphism and finish it off!

(11) Let F be a free group on $d \geq 2$ generators $x_1, \ldots, x_d$. From the above F/F' (F' is the normal subgroup generated by all commutators) is the free abelian group on $x_1, \ldots, x_d$. Use this to show that $x_1^{a_1}\cdots x_d^{a_d}$ form a Schreier transversal to F'. Then use the argument above to show that F' is a free group on a countably infinite set of generators. Conclude that every noncyclic free group contains a free group on infinitely many generators.