Assignment 1: Due Jan. 31, 2019

(1) Let X be a set and let $V \subseteq k\langle X\rangle$ denote the k-span of the elements of X. For each $i \geq 1$, let V^i denote the k-span of all products of i elements of V. Show that
$$k\langle X\rangle = k \oplus V \oplus V^2 \oplus V^3 \oplus \cdots .$$

(2) A Lie algebra is a k-vector space V with a k-bilinear map $[\cdot\,,\cdot] : V \times V \to V$ (called a Lie bracket) that satisfies $[v, w] = -[w, v]$ for $v, w \in V$ and $[[v, w], u] = [[v, u], w] + [v, [w, u]]$ (the Jacobi identity) for all $v, w, u \in V$. Show that if R is an associative algebra then $[r, s] := rs - sr$ is a Lie bracket and makes R a Lie algebra.

(3) Let X be a set with order $<$, and suppose that we extend $<$ to a degree lexicographic order on X^* and let $V \subseteq k\langle X\rangle$ denote the k-span of the elements of X. Suppose that V has a Lie bracket $[\cdot\,,\cdot]$. Show that the set S given by $\{yx - xy - [x, y] \colon y > x, \text{ with } y, x \in X\}$ is a Gröbner-Shirshov basis with respect to $<$.

(4) Let X be a set with order $<$, and suppose that we extend $<$ to a degree lexicographic order on X^* and let $V \subseteq k\langle X\rangle$ denote the k-span of the elements of X. Suppose that V has a Lie bracket $[\cdot\,,\cdot]$. Show that if I Is the ideal generated by $\{yx - xy - [x, y] \colon y > x, \text{ with } y, x \in X\}$ then $k\langle X\rangle/I$ has a basis consisting of images of monomials of the form $x_1 x_2 \cdots x_s$ with $s \geq 0$ and $x_1, \ldots, x_s \in X$ with $x_1 \leq x_2 \leq \cdots \leq x_s$.

(5) Let $X = \{A, C, T, G\}$ and let $f(n)$ denote the number of words of length n over the alphabet A, C, T, G that do not have GAT or TAG as subwords. Find a linear recurrence satisfied by $f(n)$.

(6) Let k be a field and let $R = k[x, y]/I$ (so R is commutative!) where I is generated by the polynomials $x^2 + y^3 - 2$ and $x^3 + y^2 - 2$. Show that R is finite-dimensional and find its dimension.

(7) For the next few exercises, we'll work with a commutative ring $R = k[x_1, \ldots, x_d]/I$. Suppose that we put some order on $\{x_1, \ldots, x_d\}$ and use this to impose a degree lexicographic order on (commutative) monomials in $x_1, \ldots, x_d$. Let V be the image of the space $k + kx_1 + \cdots + kx_d$ in R. Show that V^n has dimension given by the size of the set of monomials m in $\mathrm{Irr}(S)$ of degree at most n.

(8) The Hilbert series of the commutative ring $R = k[x_1, \ldots, x_d]/I$ is defined as follows. One puts a degree lexicographic order on monomials in $x_1, \ldots, x_d$ and computes a Gröbner basis S for I. Then one lets T denote the set of monomials in $\mathrm{Irr}(S)$ and finally one lets $f(n)$ denote the number of monomials of length n in $\mathrm{Irr}(S)$. Then one defines the Hilbert series to be the formal power series
$$\sum_{n\geq 0} f(n)t^n \in \mathbb{Z}[[t]].$$
Show that the Hilbert series is independent of how one assigns the order on $x_1, \ldots, x_d$ when giving the degree lexicographic order. (Hint: use the preceding question).

(9) Show that the Hilbert series of the ring R is the power series expansion of a function of the form $P(t)/Q(t)$ where $P(t)$ and $Q(t)$ are integer polynomials, $Q(0) = 1$, and all the roots of $Q(t)$ are roots of unity. (Hint: first use the fact that in the commutative case that Gröbner bases are finite to show that $f(n)$ satisfies a linear recurrence. Next use this fact to show that the Hilbert series is of the form $P(t)/Q(t)$ with P and Q integer polynomials and $P(0) = 1$. Finally, show that if $f(n)$ is the coefficient of t^n in the Hilbert series of R then $f(n) \leq n^d$ and use this to show that $Q(t)$ must have all of its roots as roots of unity.