Assignment 4. Due December 5. Feel free to work with other people in the class and use google and webpages to help to learn concepts (or ask me for definitions if you'd like); please do not ask me for hints or ask people online for help, but I will give clarification about what questions mean or explain concepts that are unclear. I will post solutions later.

(1) Let $R$ be a commutative ring and let $x \in R$ be a non-zero divisor. Show that $\mathrm{Tor}_1^R(R/Rx, M) = \{m \in M : xm = 0\}$.

(2) Let $A = \mathbb{C}[x, y, z]$. Compute $\mathrm{Tor}_i^A(\mathbb{C}[x, y, z]/(x, y), \mathbb{C}[x, y, z]/(y, z))$ for $i \geq 0$.

(3) Let $R$ be a commutative ring and let $I$ and $J$ be two ideals of $R$. Show that $\mathrm{Tor}_1^R(R/I, R/J) = I \cap J/IJ$.

(4) Use the last two exercises to show that if $C_1$ and $C_2$ are two lines in $\mathbb{C}^3$ and $I$ and $J$ denote respectively the set of polynomials in $\mathbb{C}[x, y, z]$ that vanish on all of $C_1$ and all of $C_2$ then show that $C_1$ and $C_2$ intersect non-trivially if and only if $\mathrm{Tor}_1^{\mathbb{C}[x,y,z]}(\mathbb{C}[x, y, z]/I, \mathbb{C}[x, y, z]/J)$ is nonzero. This is part of a more general principle that vanishing of higher Tor groups corresponds geometrically to subvarieties intersecting "as expected".

(5) (Betti numbers) Let $(R, M)$ be a commutative noetherian local ring. Let $A$ be a finitely generated $R$-module. Show that $A$ has a free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

such that for all $i$, $F_i$ is finite rank and $\phi_i : F_i \to F_{i-1}$ satisfies $\mathrm{Im}(\phi_i) \subseteq MF_{i-1}$. Show that $\mathrm{Tor}_i(R/M, A) = (R/M)^{b_i}$ where $b_i$ is the rank of $F_i$. The $b_i$ are called the *Betti numbers* of $A$.

(6) Compute $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for $i \geq 0$ and $n, m \geq 0$.

(7) Show that a finitely generated abelian group $A$ is free if and only if $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$.

(8) (Group cohomology) Let $G$ be a group and let $A$ be an abelian group (written additively) with a $G$-action $\cdot : G \times A \to A$ (we call this a $G$-module). Notice that $G$-modules form a category. Then we have a functor $I$ from $G$-modules to Abelian groups given by $I(A) = \{a \in A : g \cdot a = a \ \forall g \in G\}$ (the invariants). Then show that this functor is left exact but not right exact in general. We write $H^i(G, \cdot)$ for the $i$-th right derived functor of $I$. Then $H^0(G, A) = I(A)$. Show that $H^1(G, A)$ is the set of all maps (called crossed homomorphisms) $f : G \to A$ satisfying $f(xy) = f(x) + x \cdot f(y)$ modulo crossed homomorphisms of the form $f_a : G \to A$ given by $f_a(x) = x \cdot a - a$, where $a \in A$. In particular, if $G$ acts trivially on $A$ then $H^1$ just gives the homomorphisms from $G$ to $A$.

(9) Let $G$ be a finite abelian group and let $G$ act on $\mathbb{Z}$ trivially. Show that $H^2(G, \mathbb{Z})$ is isomorphic to $G$. (Hint: you could try to use the long exact sequence from cohomology, or you could just compute it, I guess.)

(10) An interesting case of group cohomology is when $G$ is when $G$ is a Galois group of $L/K$ and $A$ is the field $L$ or $L^*$ or something related. Let $L = \mathbb{Q}(i)$ and let $G = \mathbb{Z}/2\mathbb{Z}$ act on $L^*$ via conjugation. Then show that $H^1(G, L^*)$ is trivial.