Assignment 3. Due November 19 in class. Feel free to work with other people in the class and use google and webpages to help to learn concepts (or ask me for definitions if you'd like); please do not ask me for hints or ask people online for help, but I will give clarification about what questions mean or explain concepts that are unclear. I will post solutions later.

(1) Let R be a ring with identity and let $S = M_n(R)$. Show that R-Mod and S-Mod are equivalent categories. (Hint: Find a projective R-module P so that $S = \mathrm{End}_R(P)^{\mathrm{op}}$ and show that this gives an embedding of R-Mod into S-Mod.)

(2) Let $\mathcal{A}$ be a preabelian category. Show that if all monomorphisms are normal and all epimorphisms are normal in $\mathcal{A}$ then every map $f : M \to N$ factors as $f = v \circ u$ with $u : M \to \mathrm{Im}(f)$ an epimorphism and $v : \mathrm{Im}(f) \to N$ a monomorphism. Does the converse hold?

(3) Show that if P is a projective generator (that is $\mathrm{Hom}(P, -)$ is exact and faithful) in a small abelian category $\mathcal{A}$ then $P' := \amalg_I P$ is a projective generator, where I is the disjoint union of all $\mathrm{Hom}(P, A)$ as A ranges over the objects of $\mathcal{A}$ and show that for each object A there is an epimorphism from P' to A.

(4) Show that in abelian category $\mathcal{A}$ if M is an object then the functor $\mathrm{Hom}(M, -)$ is a left exact functor from $\mathcal{A}$ to Ab. Similarly, show that if $A \to B \to C \to 0$ is exact then $0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M)$ is exact.

(5) Let R be a noetherian commutative ring. Show that if P is a finitely generated projective module then the map $f : \mathrm{Spec}(R) \to \mathbb{Z}$ given by $\mathfrak{P} \mapsto \mathrm{rank}(P_{\mathfrak{P}})$ is a continuous map (here, $\mathrm{Spec}(R)$ has the Zariski topology and $\mathbb{Z}$ has the discrete topology). Recall from 446 that $\mathrm{Spec}(R)$ is connected if and only if R has 0 and 1 as its only idempotents. Conclude that in this case a projective module has a well-defined rank.

(6) Let $\mathcal{A}$ be a small abelian category and for each object A let h_A be the functor $\mathrm{Hom}(A, -) : \mathcal{A} \to \mathrm{Ab}$. Show that $\amalg_{A \in \mathrm{Ob}(\mathcal{A})} h_A$ is a projective generator for the functor category $\mathrm{Funct}(\mathcal{A}, \mathrm{Ab})$.

(7) (Noncommutative localization—for more on noncommutative localization and topology see Andrew Ranicki's book) Let R be a (not necessarily commutative) ring and let S be a multiplicatively closed subset of R. We'd like to be able to localize and form a quotient ring $S^{-1}R$. Intuitively we could just look at quotients $s^{-1}r$ but now since R is not commutative we need a mechanism for writing $(s_1^{-1}r_1)(s_2^{-1}r_2)$ in that form. Fortunately, there is a condition that ensures we can do this: the Ore condition. We say that S is left Ore if for all $a \in R$ and all $s \in S$ the intersection $Sa \cap Rs$ is nonempty. We can define right Ore analogously. Notice now that we can make sense of $S^{-1}R$ if we have $Rs_2 \cap Sr_1$ is nonempty: there is some solution $r_3 s_2 = s_3 r_1$ (we think of this as saying $s_3^{-1} r_3 = r_1 s_2^{-1}$. So now we have: $(s_1^{-1}r_1)(s_2^{-1}r_2) = s_1^{-1}(r_1 s_2^{-1}) r_2 = s_1^{-1} s_3^{-1} r_3 r_2 = (s_3 s_1)^{-1} r_3 r_2$. Similarly, if S is right Ore then we have RS^{-1}, a right localization. Now suppose that S is left and right Ore. Then show that $S^{-1}R$ is a right R-module (well, even more but this is all we need) and left $S^{-1}R$-module and that $S^{-1}R$ is flat; i.e., if $M' \to M$ is an injective homomorphism of left R-modules then the induced **map $S^{-1}R \otimes_R M' \to S^{-1}R \otimes_R M$ is injective**. (Notice that the bimodule structure gives that this tensor product has a structure of a left $S^{-1}R$ module.)

(8) (The five lemma! Get ready to diagram chase.) Suppose that R is a ring (not necessarily commutative) and suppose we have a commutative diagram of left R-modules whose rows are exact:

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\
\downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{i'} & E'
\end{array}$$

Show that if b and d are isomorphisms and a is an epimorphism and e is a monomorphism then c is an isomorphism.

(9) Let R be a commutative ring and let S be an R-algebra. Show that if M and N are R-modules, then there is an S-module homomorphism

$$h : S \otimes_R \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$$

which sends $1 \otimes f \in S \otimes_R \mathrm{Hom}_R(M, N)$ to $\mathrm{id}_S \otimes f : S \otimes M \to S \otimes N$. Show that if S is a flat R-module and M is finitely presented then this homomorphism h is in fact an isomorphism. (Hint: In the S flat, M finitely presented case, first prove the result when $M = R$; then use this to prove the result when $M = R^n$ for $n \geq 1$. Next if M is finitely presented then we have an exact sequence $R^m \to R^n \to M \to 0$ now use question 4, applying the contravariant functor $\mathrm{Hom}_R(-, N)$ and the contravariant functor $\mathrm{Hom}_S(-, N \otimes_R S)$ separately to get to left exact sequences. Now put one on top of the other use your h's as vertical maps and add an extra $0 \to$ to the left of reach row. Now use the Five-Lemma!

(10) Use question (7) to show that if R is commutative, S is multiplicatively closed, and M is finitely presented then $S^{-1}\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$.