Assignment 2. Due October 24 in class. Feel free to work with other people in the class and use google and webpages to help to learn concepts (or ask me for definitions if you'd like); please do not ask me for hints or ask people online for help, but I will give clarification about what questions mean or explain concepts that are unclear. I will post solutions later.

(1) Let R be a commutative ring. Show that every R-module M is a filtered colimit of its finitely generated submodules (Here, one takes the category of all finitely generated submodules of M with maps between them being the inclusions.)
(2) Let R be a commutative ring. Show that every R-module is a filtered colimit of finitely presented modules (finitely presented just means that it is isomorphic to R^n/N for some natural number $n \geq 1$ and some finitely generated submodule N of R^n).
(3) Let $\mathcal{B}$ be a small filtered subcategory of the category of commutative k-algebras (k a field). Show that if all objects of $\mathcal{B}$ are integral domains then the colimit is an integral domain. What if we work in the category of noncommutative (but associative) k-algebras instead?
(4) (Completion of a ring) Let R be a local noetherian ring with maximal ideal P. Then we have a subcategory of the category of rings whose objects are R/P^n for $n \geq 1$ and where we have the "reduction" map $R/P^n \to R/P^m$ for $n \geq m$. We define $\widehat{R}$ to be the limit of this diagram. Show that the cone coming from the maps R/P^n gives us a map $R \to \widehat{R}$ and that this map is injective. (Hint: you probably need Nakayama's lemma.)
(5) Let R and $\widehat{R}$ be as in the preceding question. Suppose that R is a K-algebra that is a discrete valuation ring. Show that if the composition $K \to R \to R/P$ is an isomorphism then $\widehat{R} \cong K[[t]]$, the ring of power series in one variable.
(6) Show that if p is prime and $R = \{a/b \colon p \not| b\} \subseteq \mathbb{Q}$ then R is a local ring and the completion is isomorphic to $\mathbb{Z}_p$. Why does this not contradict the preceding exercise?
(7) Let R be a ring and let M be an R-module. Show that if M is a submodule of Q then there is a maximal submodule E of Q such that $M \cap N$ is nonzero for every nonzero submodule N of E.
(8) if $\mathcal{C}$ is a category and $\mathcal{H}$ is a class of morphisms in $\mathcal{C}$ then an object I of $\mathcal{C}$ is $\mathcal{H}$-injective whenever $f : A \to I$ is a morphism and $h : A \to B$ is a morphism in $\mathcal{H}$ we have $f = g \circ h$ for some $g : B \to I$. If $\mathcal{H}$ is not specified, one takes it to be the class of monomorphisms and we just say that I is injective. Let R, M, E, and Q be as in the preceding exercise. Show that if Q is an injective R-module then so is E.
(9) Show that an injective abelian group A is divisible; that is, if $x \in A$ is nonzero and n is a nonzero integer then there is some $y \in A$ such that $x = ny$.
(10) We say that an injective module is decomposable if we have $I \cong I' \oplus I''$ with I' and I'' nonzero—otherwise, it is indecomposable. Show that any indecomposable injective abelian group is isomorphic to either $\mathbb{Q}$ or to the subgroup of $\mathbb{C}^*$ given by $\{\exp(2\pi i j/p^n) \colon n \geq 1, j \geq 0\}$ with p a prime number.