Assignment 1. Due September 24 in class. Feel free to work with other people in the class and use google and webpages to help to learn concepts (or ask me for definitions if you'd like); please do not ask me for hints or ask people online for help, but I will give clarification about what questions mean or explain concepts that are unclear. I will post solutions later.

(1) For us, we will generally restrict ourselves to categories where the Morphisms between objects form a set and we will impose this restriction throughout since it will be convenient for us in doing homological algebra. In general this is called a *locally small category* (a category where the objects and morphisms are sets is called a *small category*. We will work outside of this framework for this question. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Show that we can make a category $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ (which might not be locally small) by declaring that the objects are functors $F : \mathcal{C} \to \mathcal{D}$ and the morphisms between functors are natural transformations.

(2) (Product categories and bifunctors) Given two categories $\mathcal{C}$ and $\mathcal{D}$ show that we can make a category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs (A, B) with A an object of $\mathcal{C}$ and B an object of $\mathcal{D}$ and morphisms $(A, B) \to (A', B')$ are pairs (f, g) of morphisms with $f : A \to A'$ and $g : B \to B'$ and $\mathrm{id}_{(A,B)} = (\mathrm{id}_A, \mathrm{id}_B)$. A functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is called a *bifunctor*.

(3) Show that if R is a commutative ring then the category of R-Mod of R-modules has the property that tensor product $\otimes$ gives a bifunctor from R-Mod $\times$ R-Mod to R-Mod. This is an example of what is called a tensor category.

(4) Show that the bifunctor $\otimes$ from the preceding question is associative up to natural isomorphism and that the R-module R is a left and right identity for $\otimes$ up to natural isomorphism. (What does all this mean? Consider a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Consider the category $\mathcal{B} = \mathcal{C} \times \mathcal{C} \times \mathcal{C}$. Then we have functors $F_1 : \mathcal{B} \to \mathcal{C} \times \mathcal{C}$ given by $F_1 = (\otimes, \mathrm{id})$ and $F_2 : \mathcal{B} \to \mathcal{C} \times \mathcal{C}$ given by $F_2 = (\mathrm{id}, \otimes)$. Then $\otimes \circ F_1$ and $\otimes \circ F_2$ should be naturally isomorphic functors (i.e., there are inverse natural transformations between the functors F_1 and F_2)—that is associativity up to natural isomorphism. If we have an object I that is a left and right identity object up to natural isomorphism, this means that the functor $L_I : \mathcal{C} \to \mathcal{C}$ that sends an object C to $I \otimes C$ and a morphism $f : C \to D$ to $L_I(f) : I \otimes C \to I \otimes D$ given by $\mathrm{id}_I \otimes f$ should be natural isomorphic to the identity functor; similarly, the functor R_I which sends C to $C \otimes I$ should be naturally isomorphic to the identity functor.)

(5) Let $\mathcal{G}$ be the category of groups and let $\mathcal{A}$ be the category of abelian groups. Let $R : \mathcal{G} \to \mathcal{A}$ be the functor described in class given by $R(G) = G/G'$ and let $S : \mathcal{A} \to \mathcal{G}$ be the "forgetful" functor given by $S(A) = A$ (where we forget the extra abelian constraint). Answer all of the following questions:
- is S a left adjoint for R?
- is S a right adjoint for R?
- does S have a left adjoint?
- does S have a right adjoint?
- does R have a left adjoint?
- does R have a right adjoint?

(6) Let $\mathcal{C}$ be the category with one object given by the set $\{1, 2\}$ and two morphisms: the identity and the transposition. As in question 1, we have an identity functor $I : \mathcal{C} \to \mathcal{C}$ and we have another functor $F : \mathcal{C} \to \mathcal{C}$ that sends every morphism to the identity map. Show that F and I are not naturally isomorphic despite behaving identically on objects.

(7) (Groupoids!) Let $\mathcal{C}$ be a small category (i.e., the objects (and morphisms) are sets). We say that $\mathcal{C}$ is a *groupoid* if every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ has an inverse; i.e., there is some $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. Show that every group (even infinite ones) can be realized as a groupoid with a single object. (Think of the above question as a hint.)

(8) Let $\mathcal{C}$ be a groupoid and suppose that for all objects A, B the set of homomorphisms from A to B is either empty or consists of a single homomorphism. Show that this structure gives an equivalence relation on the set of objects of $\mathcal{C}$ and conversely that an equivalence relation on a set gives a groupoid whose objects are precisely the elements of the set.

(9) Let X be a topological space and let $\mathcal{P}$ be the category whose objects are the points of X and where the morphisms from x to y (with $x, y \in X$) are precisely the continuous maps $h : [0,1] \to X$ with $h(0) = x$ and $h(1) = y$ (we call these paths from x to y). As before we say two paths $f, g : [0,1] \to X$ with $f(0) = g(0) = x$ and $f(1) = g(1) = y$ are homotopy equivalent if there is a continuous map $H : [0,1]^2 \to X$ with $H(0,t) = x$ for all t and $H(1,t) = y$ for all t and $H(s,0) = f(s)$ and $H(s,1) = g(s)$. Show that homotopy equivalence is an equivalence relation and define $\mathrm{Hom}_{\mathcal{P}}(x,y)$ to be the set of (homotopy) equivalence classes of continuous paths from x to y. Show that $\mathcal{P}$ is a groupoid. This is called the fundamental groupoid of X.

(10) (Stone-Cech compactification as a left adjoint) Given a topological space X, let $C(X)$ denote the collection of continuous maps from X to $[0,1]$. Let $Y = [0,1]^{C(X)}$. Then Y has a product topology (you can ask me or look this up if you do not know it). Then Y is compact since $[0,1]$ is compact and we can then apply Tychonoff's theorem to get compactness; Y is also Hausdorff. Then we have a map $\phi : X \to Y$, given by $\phi(x) = (f(x))_{f \in C(X)}$; that is, each coordinate corresponds to some continuous map $f \in C(X)$ and we send x to $f(x)$ in that coordinate. We then define $\beta(X)$ to be the closure of $\phi(X)$ in Y. Then $\beta(X)$ is compact and Hausdorff as it is a closed subset of a compact Hausdorff space. Then we have a map $\phi : X \to \beta(X)$. Show that ϕ is continuous. Now let $\mathcal{T}$ be the category of topological spaces and let $\mathcal{C}$ be the category of compact Hausdorff spaces. Show that we have the following universal property: if X is a topological space and C is compact Hausdorff and $f : X \to C$ is continuous then there is a unique continuous map $\beta f : \beta(X) \to C$ such that $(\beta f) \circ \phi = f$. Now show that there is a functor $F : \mathcal{T} \to \mathcal{C}$ given by $F(X) = \beta(X)$ and if $f : X \to Y$ show using the universal property that we have a continuous map $F(f) : \beta(X) \to \beta(Y)$ that makes F into a functor.

(11) Show in the preceding question that F is the left adjoint for the functor $G : \mathcal{C} \to \mathcal{T}$ which is just the inclusion functor.